Design and Analysis of Real-Time Systems SS 2013

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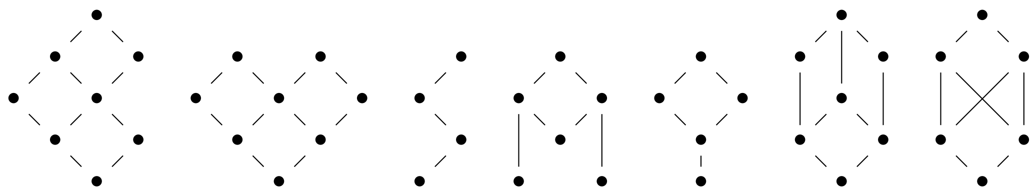
Deadline: Friday, May 10, 2013, 14:15
Please bring your submission to room 402 (in building E1.3), or send it via e-mail to abel@cdl.uni-saarland.de.

Assignment 2

Problem 1: Complete Lattices (7 Points)

Which of the following Hasse diagrams represent complete lattices? Can you "fix" the diagrams that do not represent complete lattices (by adding or removing elements)?



(a) (b) (c) (d) (e) (f) (g)

Problem 2: Monotone functions (3 Points)

Find an order such that $F(X) = \{f(x) \mid x \in X\}$ is a monotone function. Give a proof for your answer.

Problem 3: Kleene Iteration (4+2 Points)

You are given the partial order $(\mathcal{P}(\mathbb{N}), \subseteq)$.

1. Let $f : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N}), f(X) = \{(n \bmod 4) + 5 \mid n \in \mathbb{N} \land n \leq |X|\}$, where $|X|$ denotes the number of elements in the set X. Use Kleene Iteration to compute the least fixed point of f.
2. Does Kleene Iteration terminate for all monotone functions in $\mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$? If not, describe one function, where it does not terminate.

Problem 4: Ascending Chain Condition (8 Points)

Do the following partially ordered sets have infinite ascending chains? Justify your answers.

1. $(\mathbb{N}, \leq)$
2. $(\mathbb{N} \cup \{\infty\}, \leq)$
3. $(\mathcal{P}(\{a, b, c, d\}), \subseteq)$
4. $(\mathbb{N} \cup \{\bot, \top\}, \preceq)$, where $a \preceq b :\Leftrightarrow a = b \vee a = \bot \vee b = \top$

Problem 5: Fixed Point Theorem (4 Points)

Prove the fixed point theorem from the lecture: Let $(S, \leq)$ be a complete lattice that satisfies the ascending chain condition, and let $f : S \to S$ be a monotone function. Then, there is an $n \in \mathbb{N}$, such that $lfp(f) = f^n(\bot)$.

You can proceed as follows:

1. Prove that there is an $n \in \mathbb{N}$ such that $f^n(\bot) = f^{n+1}(\bot)$.
2. Prove that for this n, $f^n(\bot) = lfp(f)$.