Verification of Real-Time Systems

Numerical Abstractions

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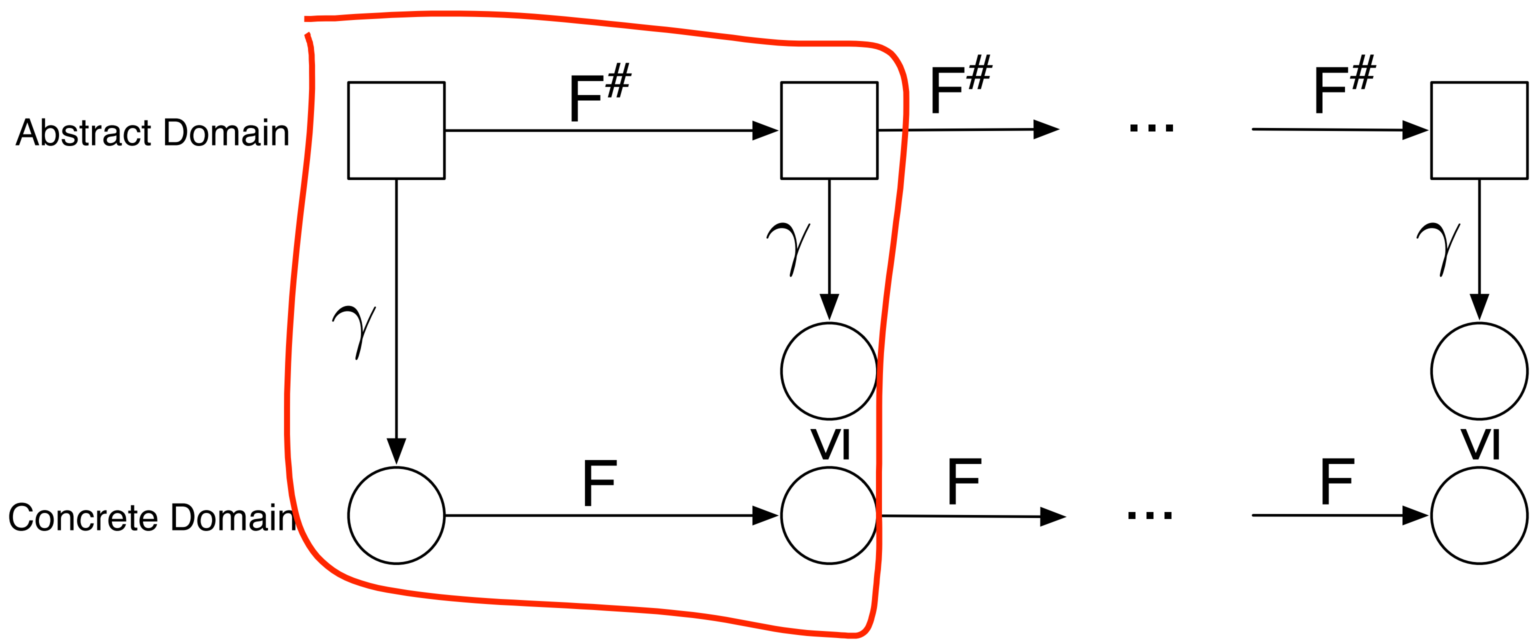
Advanced Lecture, Summer 2015

Recap: From Local to Global Correctness: Kleene Iteration



Recap: Fixpoint Transfer Theorem

Let $(L, \leq)$ and $(L^{\#}, \leq^{\#})$ be two lattices, $\gamma : L^{\#} \rightarrow L$ a monotone function, and $F : L \rightarrow L$ and $F^{\#} \rightarrow F^{\#}$ two monotone functions, with

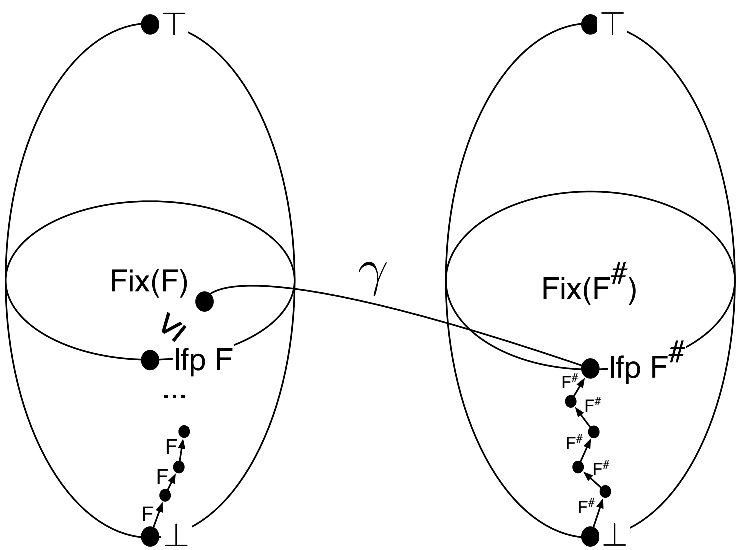
$$\forall l^{\#} \in L^{\#} : \gamma(F^{\#}(l^{\#})) \geq F(\gamma(l^{\#})).$$

Local Correctness

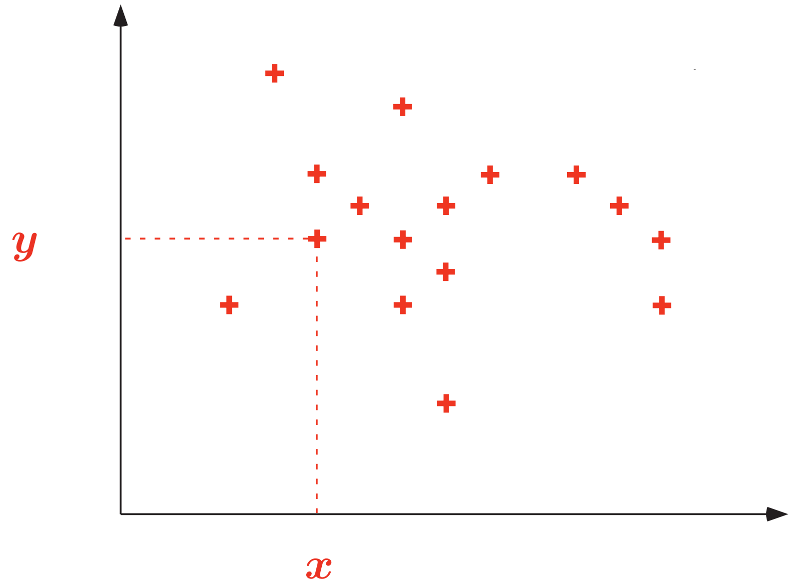
Then:

$$lfp\ F \leq \gamma(lfp\ F^{\#}).$$

Global Correctness

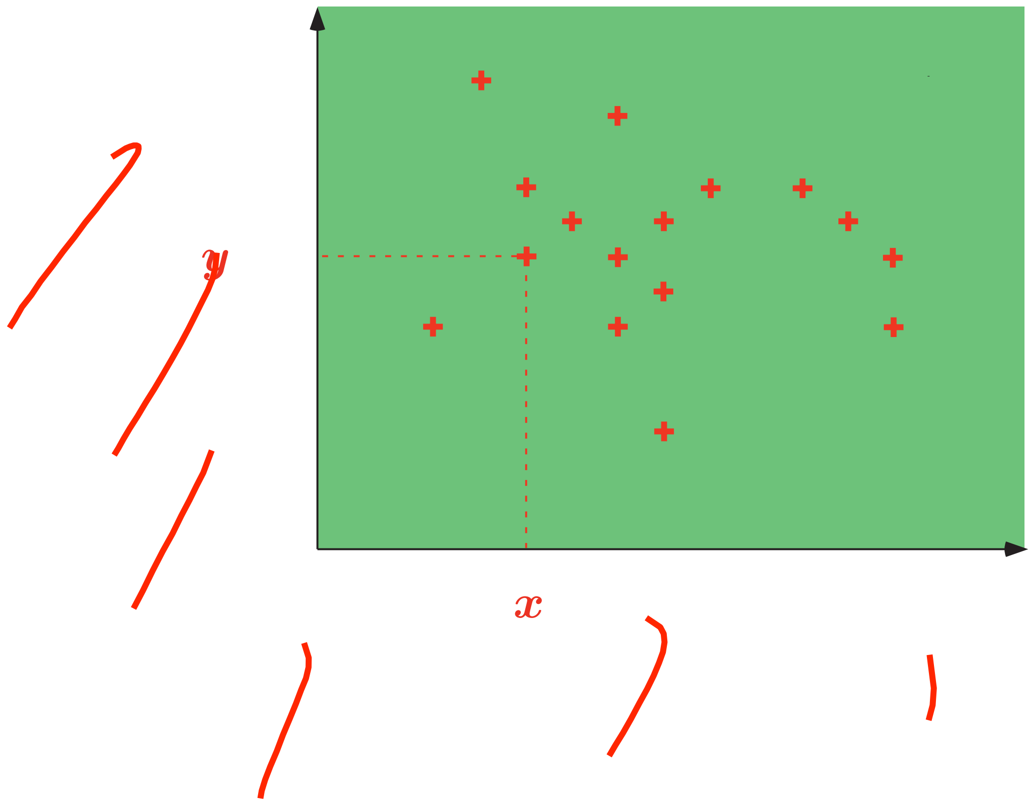


Overview: Numerical Abstractions



$$\{\ldots, \langle 19,\ 77\rangle, \ldots,\\ \langle 20,\ 03\rangle, \ldots\}$$

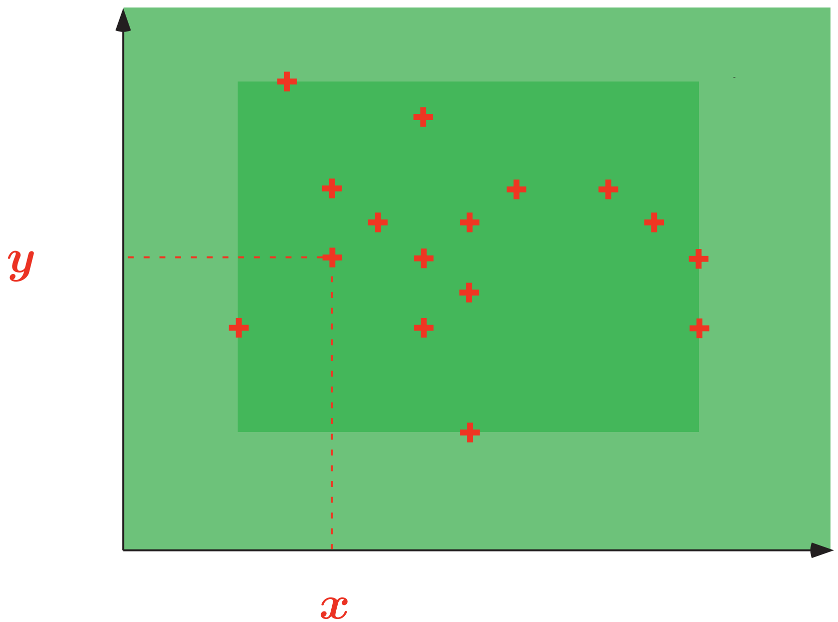
Overview: Numerical Abstractions
Signs (Cousot & Cousot, 1979)



$$\begin{cases} x \geq 0 \\ y \geq 0 \end{cases}$$

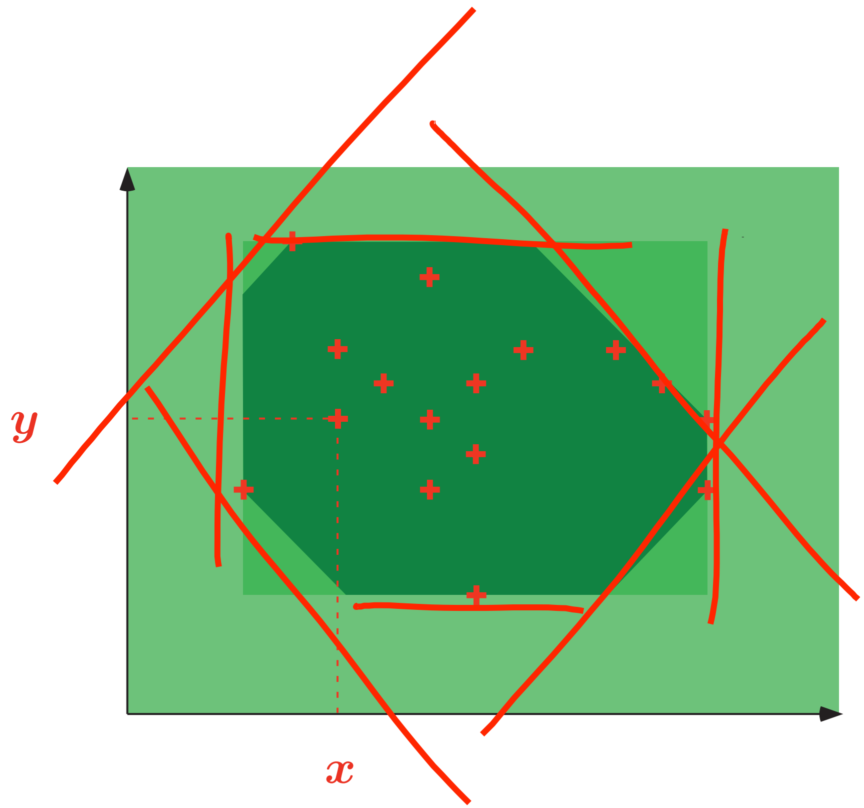
Overview: Numerical Abstractions
Intervals (Cousot & Cousot, 1976)

BOXES



$$\begin{cases} x \in [19,\ 77] \\ y \in [20,\ 03] \end{cases}$$

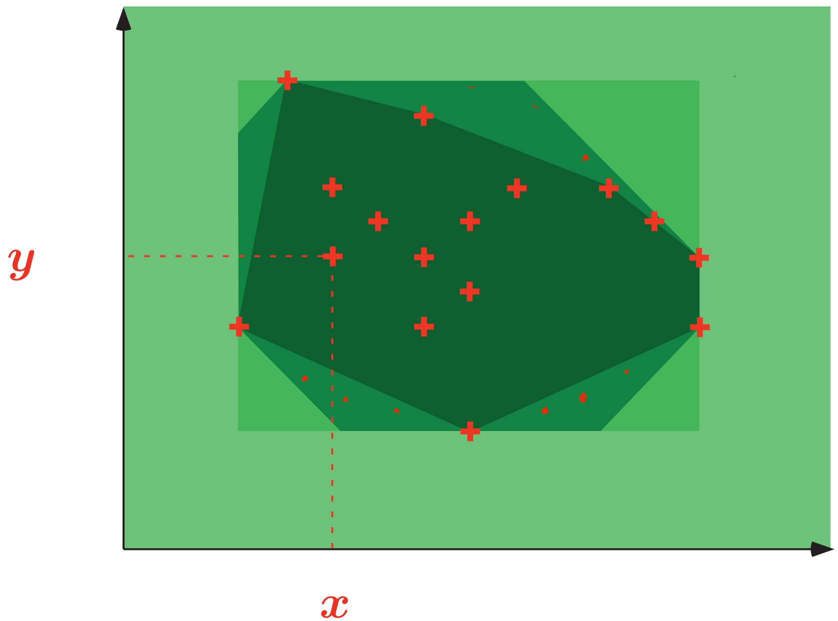
Overview: Numerical Abstractions
Octagons (Mine, 2001)



$$\begin{cases} 1 \le x \le 9 \\ x + y \le 77 \\ 1 \le y \le 9 \\ x - y \le 99 \end{cases}$$

$x \geq y$

Overview: Numerical Abstractions
Polyhedra (Cousot & Halbwachs, 1978)



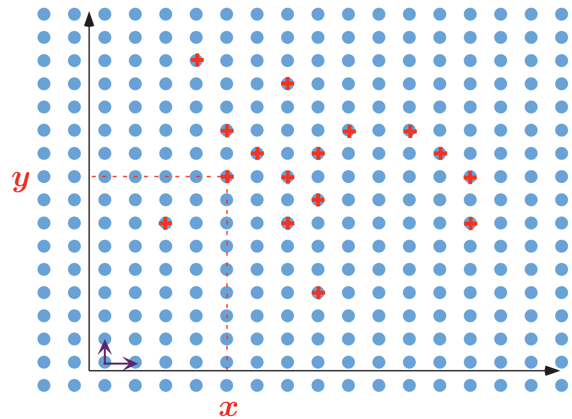
$$\begin{cases} 19x + 77y \leq 2004 \\ 20x + 03y \geq 0 \end{cases}$$

→ *Expensive…*

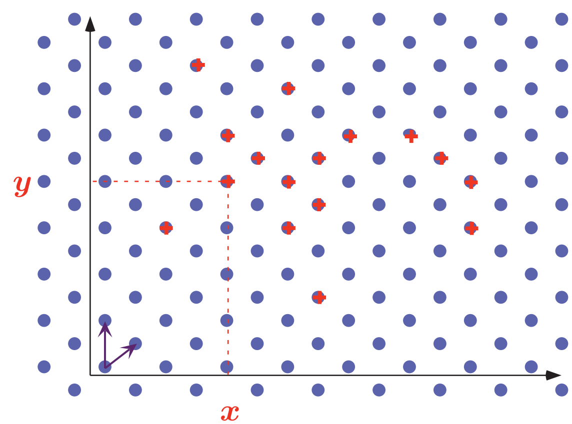
Overview: Numerical Abstractions Simple and Linear Congruences (Granger, 1989+1991)

odd ≡ 1 mod 2

even ≡ 0 mod 2



$$\begin{cases} x = 19 \bmod 77 \\ y = 20 \bmod 99 \end{cases}$$

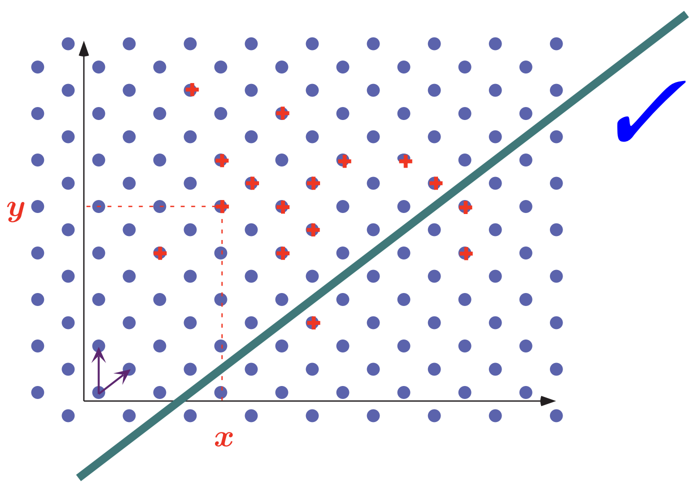


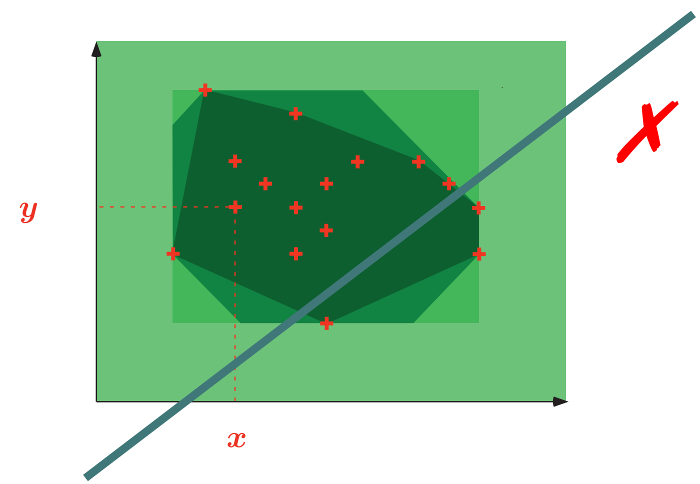
$$\begin{cases} 1x + 9y = 7 \bmod 8 \\ 2x - 1y = 9 \bmod 9 \end{cases}$$

Numerical Abstractions

Which abstraction is the most precise?

Depends on questions you want to answer!





Numerical Abstractions

Which abstraction is the most precise?

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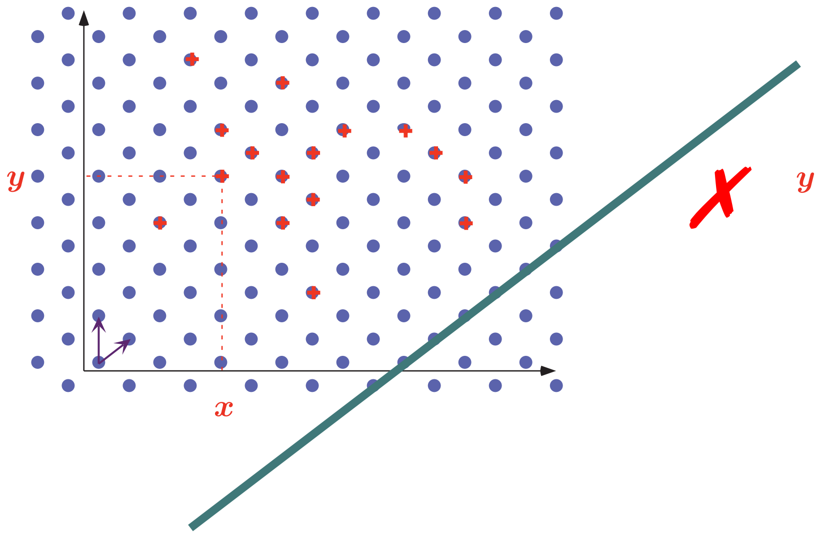
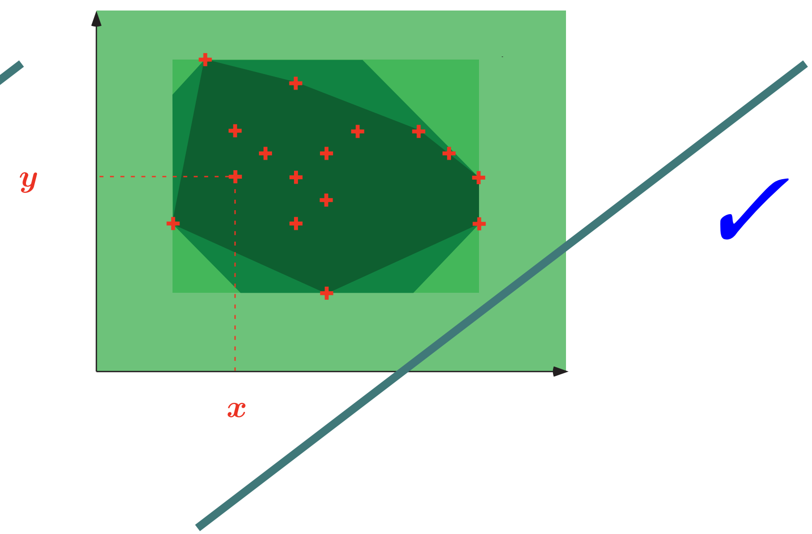
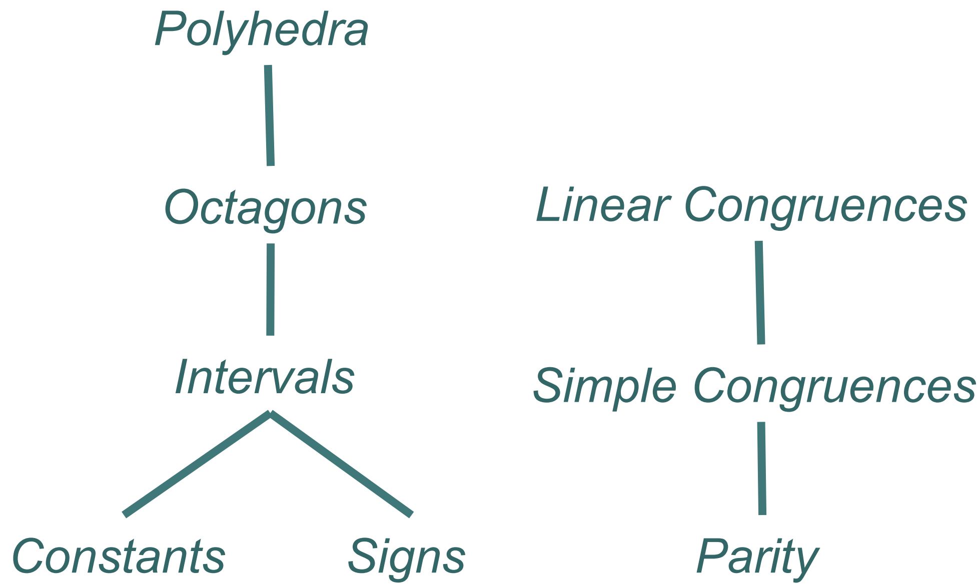
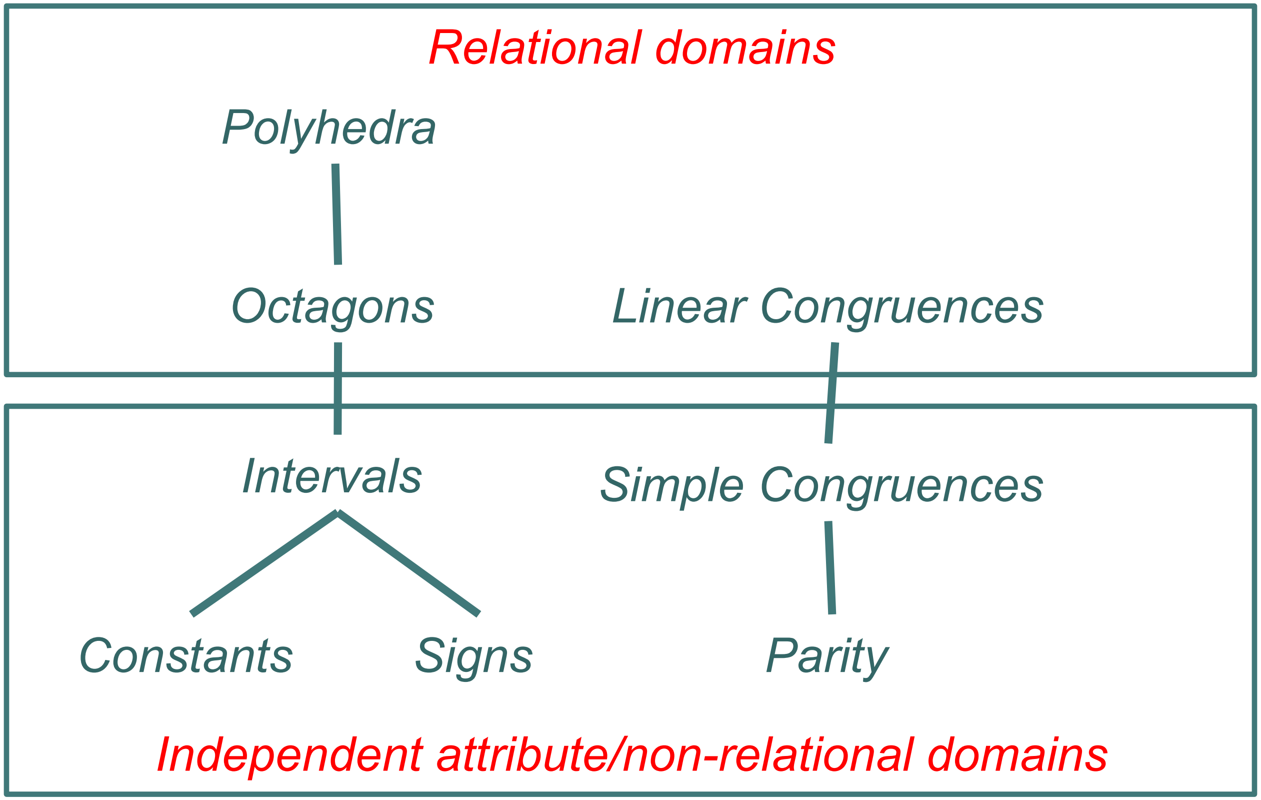




Partial Order of Abstractions



Partial Order of Abstractions



Characteristics of Non-relational Domains

- Non-relational/independent attribute abstraction:
  - Abstract each variable separately

$$(\mathcal{P}(\mathbb{Z}), \subseteq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} (\textsc{Numerical}, \sqsubseteq)$$

  - Maintains no relations between variable values
- Can be lifted to an abstraction of valuations of multiple variables in the expected way:

$$(\mathcal{P}(\mathit{Vars} \to \mathbb{Z}), \subseteq) \underset{\alpha_1}{\overset{\gamma_1}{\leftrightarrows}} (\mathit{Vars} \to \mathcal{P}(\mathbb{Z}), \leq) \underset{\alpha_2}{\overset{\gamma_2}{\leftrightarrows}} (\mathit{Vars} \to \textsc{Numerical}, \sqsubseteq)$$

$$\alpha_2(f) := \lambda x \in \mathit{Vars}.\alpha(f(x)) \qquad \gamma_2(f^{\#}) := \lambda x \in \mathit{Vars}.\gamma(f^{\#}(x))$$

The Interval Domain

Abstracts sets of values by enclosing interval

$$\textsc{Interval} = \{[l,u] \mid l \leq u, l \in \mathbb{Z} \cup \{-\infty\}, u \in \mathbb{Z} \cup \{\infty\}\} \cup \{\bot\}$$

where $\leq$ is appropriately extended from $\mathbb{Z} \times \mathbb{Z}$ to $(\mathbb{Z} \cup \{-\infty\}) \times (\mathbb{Z} \cup \{\infty\})$

Intervals are ordered by inclusion:

$$\bot \sqsubseteq x \quad \forall x \in \textsc{Interval}$$

$$[l,u] \sqsubseteq [l',u'] \; \mathit{if} \; l' \leq l \wedge u \leq u'$$

$(\textsc{Interval}, \sqsubseteq)$ forms a complete lattice.

Concretization and Abstraction of Intervals

- Concretization:

$$\gamma(\bot) = \emptyset$$

$$\gamma([l, u]) = \{n \in \mathbb{Z} \mid l \leq n \leq u\}$$

- Abstraction:

$$\alpha(\emptyset) = \bot$$

$$\alpha(S) = [\inf S, \sup S]$$

They form a Galois connection.

Interval Arithmetic

$x \cdot y = x \cdot \frac{1}{y}$

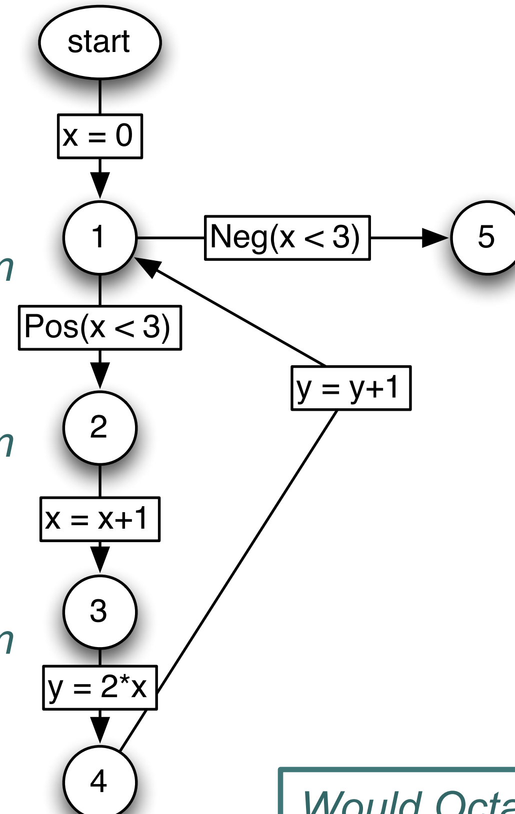
Calculating with Intervals:

$$
\begin{aligned}
[a,b] &\quad + \quad [c,d] &= \quad & [a+c, b+d] \\
[a,b] &\quad - \quad [c,d] &= \quad & [a-d, b-c] \\
[a,b] &\quad * \quad [c,d] &= \quad & [\min(ac,ad,bc,bd), \max(ac,ad,bc,bd)] \\
[a,b] &\quad / \quad [c,d] &= \quad & [a,b] * [1/d, 1/c], \underline{0 \notin [c,d]}
\end{aligned}
$$

$[a,b] * [1/d, 1]$ $\quad 0 \in [c,d],\ c < 0 < d$

$\cup\ [a,b] * [-1, 1/c]$

Example: Interval Analysis



Point				
1	x → [0,3] y → [3,7]	x → [0,2] y → [3,5]	x → [0,1] y → [3,3]	x → [0,0] y → bottom
2		x → [0,2] y → [3,5]	x → [0,1] y → [3,3]	x → [0,0] y → bottom
3		x → [1,3] y → [3,5]	x → [1,2] y → [3,3]	x → [1,1] y → bottom
4		x → [1,3] y → [2,6]	x → [1,2] y → [2,4]	x → [1,1] y → [2,2]
5	x → [3,3] y → [3,7]			

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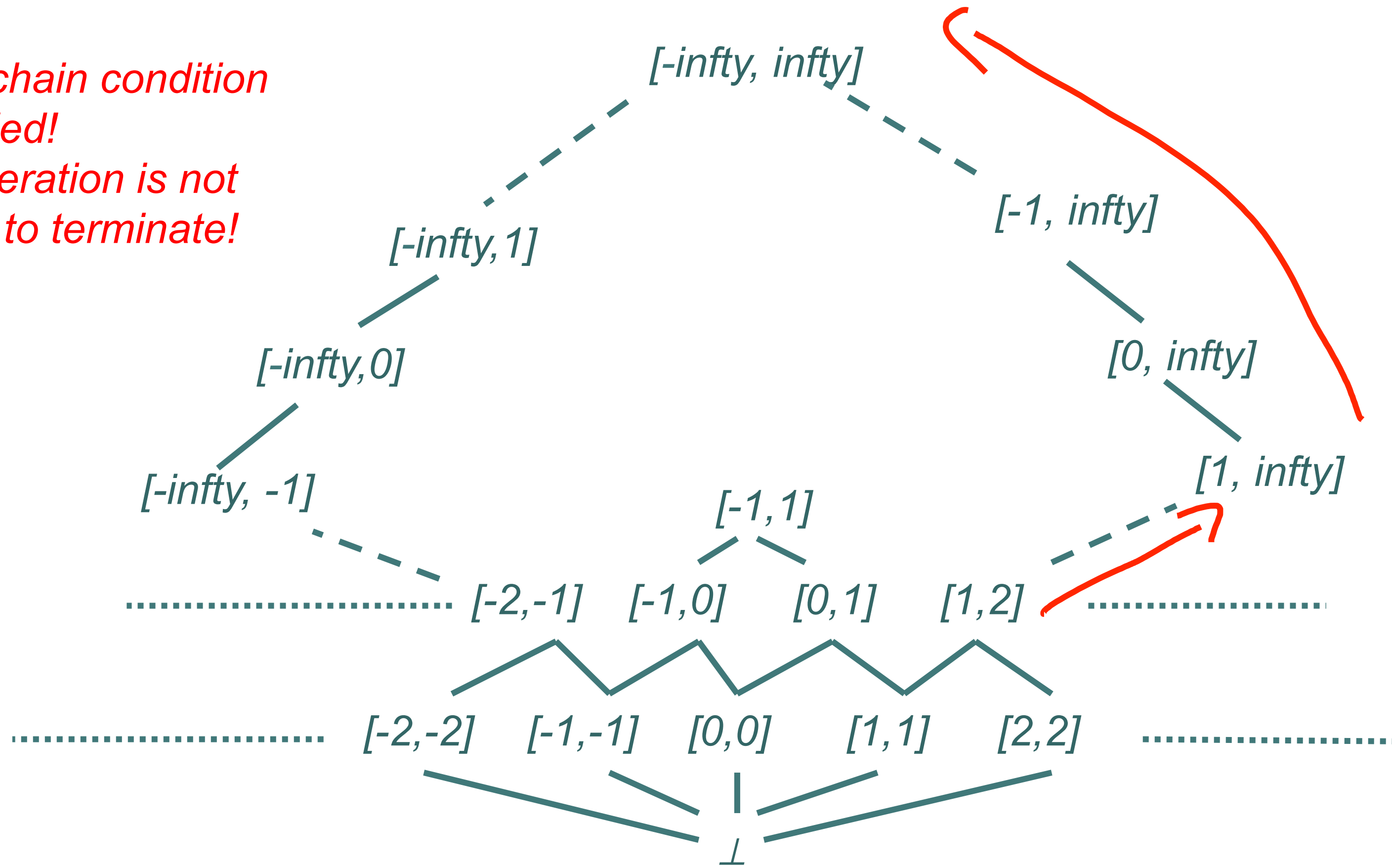
Imprecise due to non-relational analysis

Would Octagons determine that y must be 7 at program point 5?

$y = 2x + 1$

Intervals, Hasse diagram

Ascending chain condition is not satisfied!
→ Kleene iteration is not guaranteed to terminate!



Example: Interval Analysis

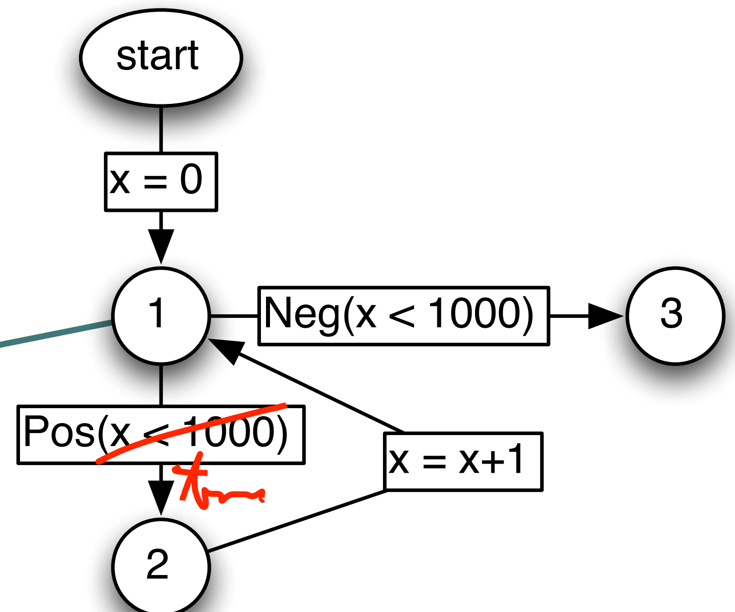
$$x \mapsto \bot$$

$$x \mapsto [0, 0]$$

$$x \mapsto [0, 1]$$

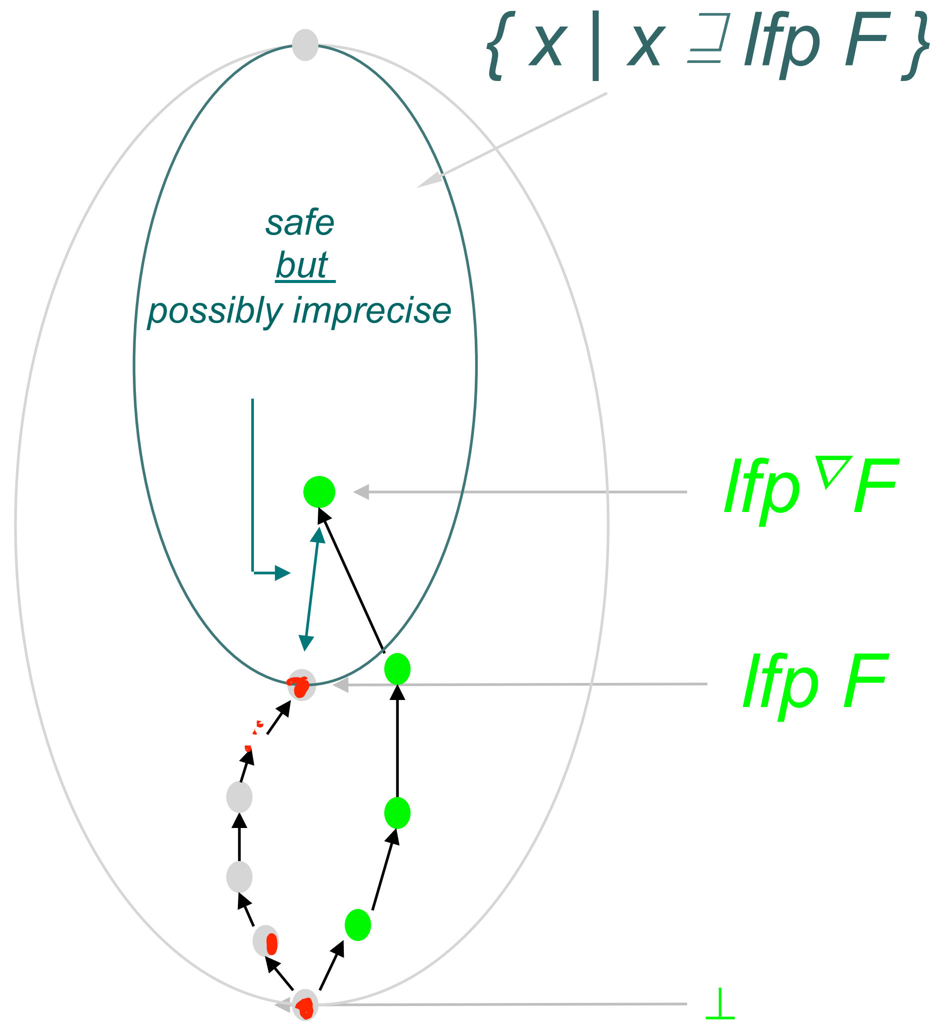
$$\ldots$$

1000 iterations later → $x \mapsto [0, 1000]$



Solution: Widening
"Enforce Ascending Chain Condition"

- Widening enforces the ascending chain condition during analysis.
- Accelerates termination by moving up the lattice more quickly.
- May yield imprecise results…



Widening: Formal Requirement

A widening ∇ is an operator $\nabla : D \times D \rightarrow D$ such that

1. Safety: $x \sqsubseteq (x \nabla y)$ and $y \sqsubseteq (x \nabla y)$ "upper bound"
2. Termination:

   forall ascending chains $x_0 \sqsubseteq x_1 \sqsubseteq \ldots$ the chain

$$y_0 = x_0$$
$$y_{i+1} = y_i \nabla x_{i+1}$$

   is finite.

Widening Operator for Intervals

Simplest solution:

$$\bot \nabla x = \; x \nabla \bot = x$$

$$[l,u]\nabla[l',u'] = \; \left[\begin{cases} l & : l' \geq l \\ -\infty & : l' < l \end{cases}, \begin{cases} u & : u' \leq u \\ \infty & : u' > u \end{cases} \right]$$

Example:

$$[3,5]\nabla[2,5] = [-\infty,5]$$

$$[3,5]\nabla[4,5] = [3,5]$$

$$[3,5]\nabla[4,6] = [3,\infty]$$

$$[3,5]\nabla[2,6] = [-\infty,\infty]$$

Example Revisited: Interval Analysis with Simple Widening

Standard Kleene Iteration:

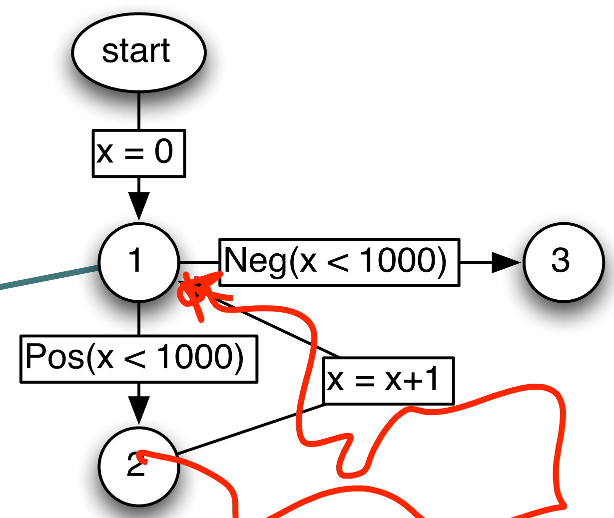
$$\bot \le F(\bot) \le F^2(\bot) \le F^3(\bot) \le \dots$$

Kleene Iteration with Widening: $F_\nabla(x) := x \nabla F(x)$

$$\bot \le F_\nabla(\bot) \le F_\nabla^2(\bot) \le F_\nabla^3(\bot) \le \dots$$

$x \mapsto [0,0]$

$x \mapsto [0,\infty]$



Do we need to apply widening at all program points?

→ *Quick termination but imprecise result!*

More Sophisticated Widening for Intervals

Define set of jump points (barriers) based on constants appearing in program, e.g.:

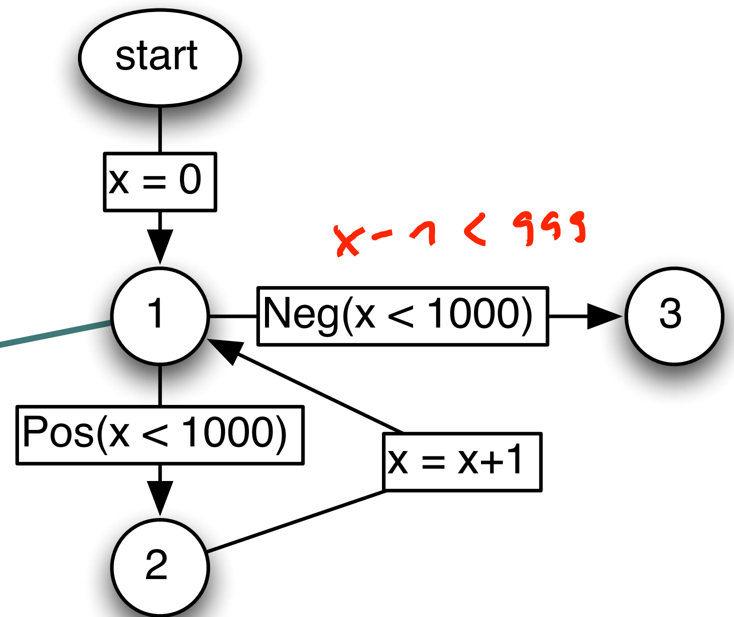
$$\mathcal{J} = \{-\infty, 0, 1, 1000, \infty\}$$

Intuition: "Don't jump to –infty, +infty immediately but only to next jump point."

$$[l, u] \nabla [l', u'] = \left[\begin{cases} l & : l' \geq l \\ \max\{x \in \mathcal{J} \mid x \leq l'\} & : l' < l \end{cases}, \begin{cases} u & : u' \leq u \\ \min\{x \in \mathcal{J} \mid x \geq u'\} & : u' > u \end{cases} \right]$$

Example Revisited:
Interval Analysis with Sophisticated Widening

$$x \mapsto [0, 0]$$
$$x \mapsto [0, 1]$$
$$x \mapsto [0, 1000]$$



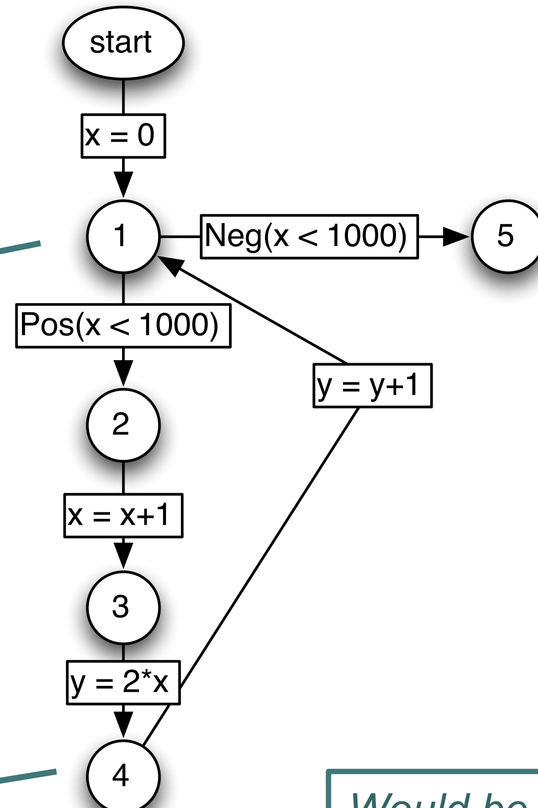
→ *More precise, potentially terminates more slowly.*

Another Example: Interval Analysis with Sophisticated Widening

$$x \mapsto [0, 0]$$

$$x \mapsto [0, 1]$$

$$x \mapsto [0, 1000]$$



$$y \mapsto [2, 2]$$

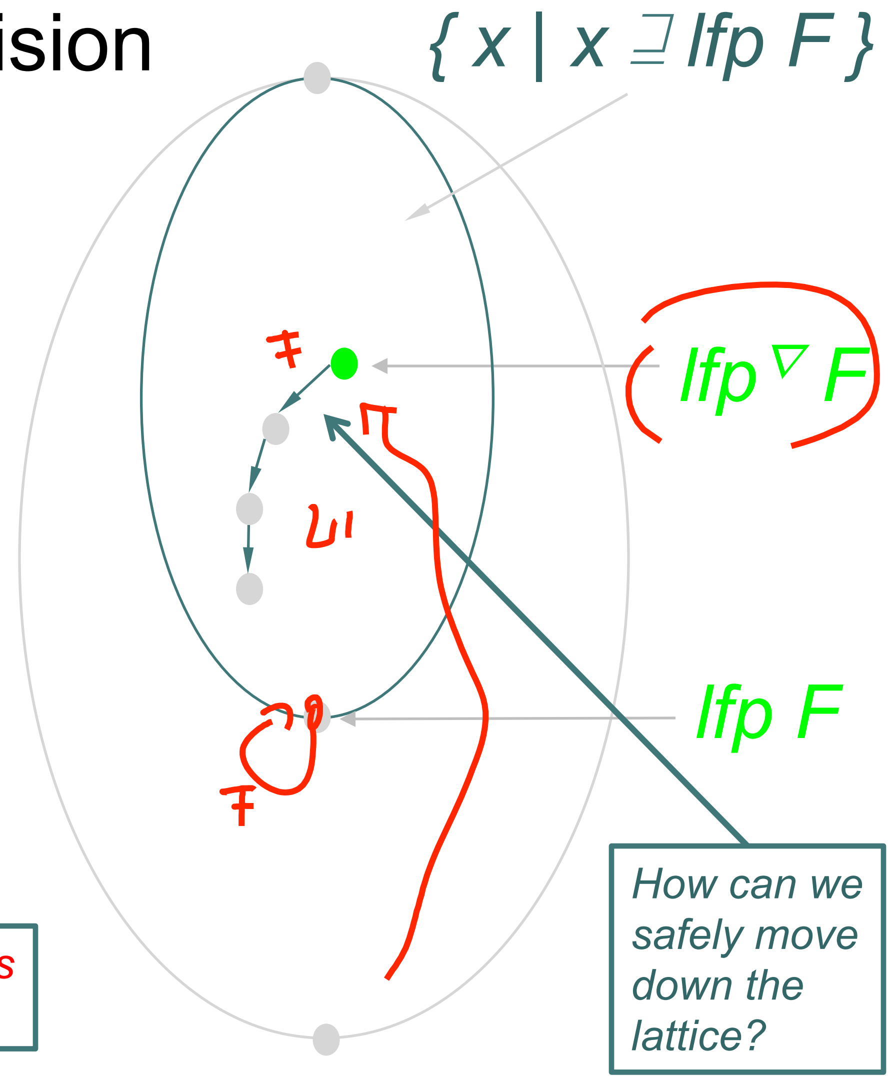
$$y \mapsto [2, 1000]$$

$$y \mapsto [2, \infty]$$

Would be [2, 2000] in least fixed point, but 2000 does not appear in the program…

Narrowing: Recovering Precision

- Widening may yield imprecise results by overshooting the least fixed point.
- Narrowing is used to approach the least fixed point from above.



Possible problem: infinite descending chains
Is it really a problem?

Narrowing: Recovering Precision

Widening terminates at a point $x \sqsupseteq \text{lfp } F$.

We can iterate:

$$x_0 = x$$
$$x_{i+1} = F(x_i) \sqcap x_i$$

Safety:

By monotonicity we know $F(x) \sqsupseteq F(\text{lfp } F) = \text{lfp } F$.
By induction we can easily show that $x_i \sqsupseteq \text{lfp } F$ for all i.

Termination:

Depends on existence of infinite descending chains.

Narrowing: Formal Requirement

A narrowing Δ is an operator $\Delta : D \times D \rightarrow D$ such that

1. Safety: $l \sqsubseteq x$ and $l \sqsubseteq y \;\rightarrow\; l \sqsubseteq (x \,\Delta\, y) \sqsubseteq x$
2. Termination:

   for all descending chains $x_0 \sqsupseteq x_1 \sqsupseteq \ldots$ the chain

   $$y_0 = x_0$$
   $$y_{i+1} = y_i \,\Delta\, x_{i+1}$$

   is finite.

Is $\sqcap$ ("meet") a narrowing operator on intervals?

Another Example Revisited: Interval Analysis with Widening and Narrowing

Result after Widening:

$x \mapsto [0,0]$

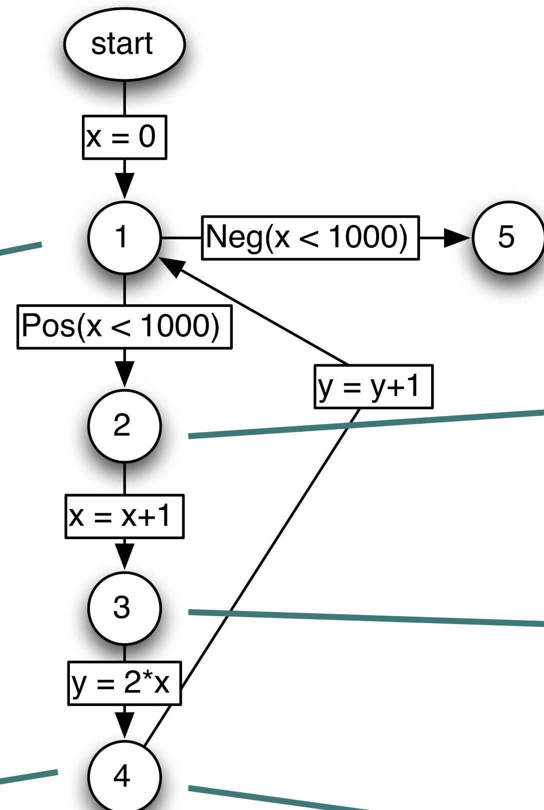
$x \mapsto [0,1]$

$x \mapsto [0,1000]$

$y \mapsto [2,2]$

$y \mapsto [2,1000]$

$y \mapsto [2,\infty]$



Result after Narrowing:

$x \mapsto [1000,1000]$

$y \mapsto [3,2001]$

$x \mapsto [0,999]$

$x \mapsto [1,1000]$

$y \mapsto [2,2000]$

→ *Precisely the least fixed point!*

Applications of Numerical Domains

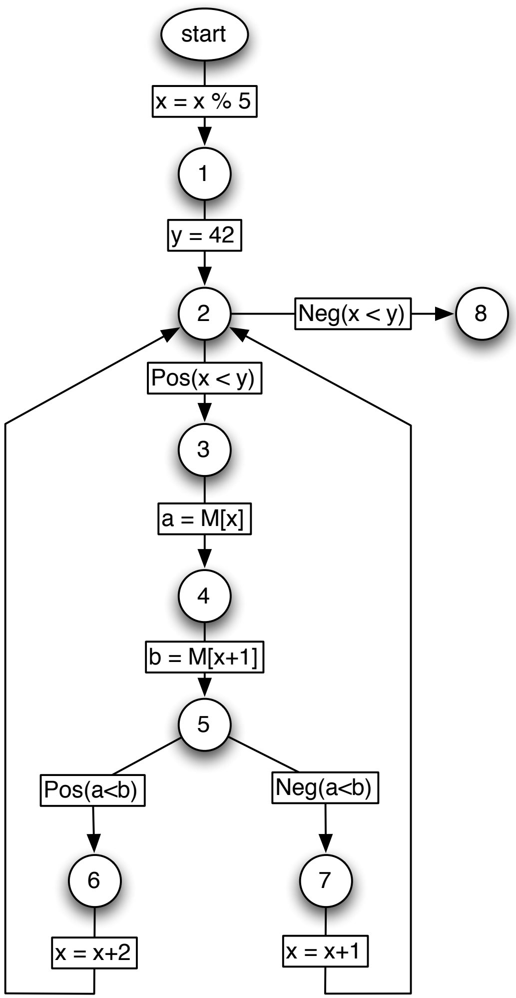
As input to other analyses:

- Cache Analysis
- Dependencies between memory accesses

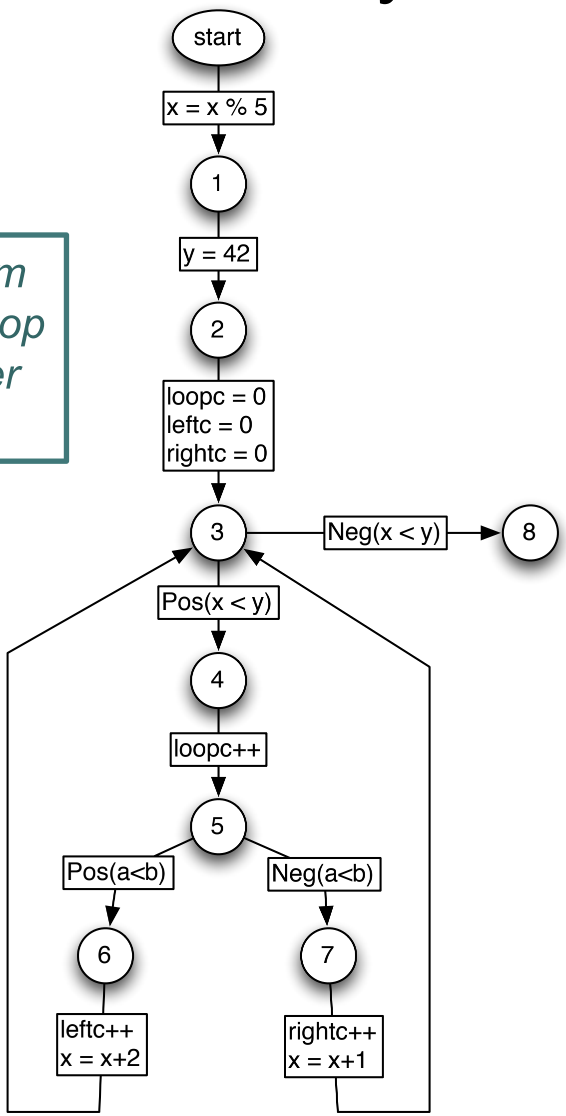
Loop Bound Analysis:

- Instrument program with loop iteration counters
- Determine maximal value of counter
- Requires relational analysis

Reduction:
Loop Bound Analysis to Value Analysis



Instrument program with counters of loop iterations and other interesting events



Summary

- Interval Analysis:

  A non-relational value analysis
- Widenings for termination in the presence of Infinite Ascending Chains
- Narrowings to recover precision
- Basic Approach to Loop Bound Analysis based on Value Analysis

Outlook

- Cache Abstractions
- Schedulability Analysis
- Cache-Related Preemption Delay
- Predictable Microarchitectures