SAARLAND UNIVERSITY
COMPUTER SCIENCE

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Verification of Real-Time Systems SS 2015

Assignment 3

Deadline: May 21, 2015, before the lecture

Exercise 3.1: Kleene Iteration (4+2 Points)

You are given the partial order $(\mathcal{P}(\mathbb{N}), \subseteq)$.

1. Let $f : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N}), f(X) = \{(n \bmod 4) + 5 \mid n \in \mathbb{N} \wedge n \leq |X|\}$, where $|X|$ denotes the number of elements in the set X. Use Kleene Iteration to compute the least fixed point of f.
2. Does Kleene Iteration terminate for all monotone functions in $\mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$? If not, describe one function, where it does not terminate.

Exercise 3.2: Ascending Chain Condition (5 Points)

Do the following partially ordered sets have infinite ascending chains? Justify your answers.

1. $(\mathbb{N}, \leq)$
2. $(\mathbb{N} \cup \{\infty\}, \leq)$
3. $(\mathbb{N}, \geq)$
4. $(\mathcal{P}(\{a, b, c, d\}), \subseteq)$
5. $(\mathbb{N} \cup \{\bot, \top\}, \preceq)$, where $a \preceq b :\Leftrightarrow a = b \vee a = \bot \vee b = \top$

Exercise 3.3: Fixed Point Theorem (4 Points)

Prove the fixed point theorem from the lecture: Let $(S, \leq)$ be a complete lattice that satisfies the ascending chain condition, and let $f : S \to S$ be a monotone function. Then, there is an $n \in \mathbb{N}$, such that $lfp(f) = f^n(\bot)$.

You can proceed as follows:

1. Prove that there is an $n \in \mathbb{N}$ such that $f^n(\bot) = f^{n+1}(\bot)$.
2. Prove that for this n, $f^n(\bot) = lfp(f)$.

Exercise 3.4: Lattice Height (1+2 Points)

Calculate the height, i.e. the length of the longest ascending chain, of the following lattices:

1. Powerset lattice $(\mathcal{P}(X), \subseteq)$ of set X,
2. Total function space lattice $A \to L$ with set A and lattice L.

Exercise 3.5: Fixed Points (4 Points)

A fixed point x of a function f is called *unique* if for every y such that $f(y) = y$, $y = x$. Prove that if there exists an $n \in \mathbb{N}^+$ such that x is a unique fixed point of f^n, then x is also a fixed point of f.