Verification of Real-Time Systems SS 2015

Assignment 2

Deadline: Thursday, May 14, 2015, 14:00 (no lecture)
Please send your submission via e-mail to sebastian.hahn@cs.uni-saarland.de, or bring it to room 402 (in building E1.3) on Wednesday, May 13, 2015, until 16:00.

Exercise 2.1: Partial Order (10 = 3+2+2+3 Points)

Prove or disprove whether the following binary relations are partial orders.

1. Reachability of vertices (reflexive, transitive closure of the edge relation) in a directed, acyclic graph $G = (V, E)$ with vertex set $V$ and edge relation $E$.

2. Reachability of vertices in a directed graph $G = (V, E)$ with vertex set $V$ and edge relation $E$.

3. Reachability of vertices in an undirected graph $G = (V, E)$ with vertex set $V$ and edge relation $E$.

4. For a set $X$ and a partially-ordered set $P$, the function space $F : X \to P$, where $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x$ in $X$. 

Exercise 2.2: Monotone Functions (13 = 3+4+3+3 Points)

1. Which of the following functions are monotone (under the natural order $\leq$)? Justify your answers.
   - $f : \mathbb{N} \to \mathbb{N}, f(x) = x + 1$
   - $g : \mathbb{N} \to \mathbb{N}, f(x) = 0$
   - $h : \mathbb{N} \to \mathbb{N}, f(x) = 5 \cdot x$
   - $i : \mathbb{N} \to \mathbb{N}, f(x) = x \mod 5$
   - $j : \mathbb{N} \to \mathbb{N}, f(x) = \lfloor x/5 \rfloor$

2. Given $A = \mathcal{P}({\mathbb{N}})$ and $B = \mathbb{N}$, choose partial orders for $A$ and $B$, and find two functions $f, g : A \to B$ such that $f$ is monotone under the chosen order and $g$ is not.

3. Let $f : P \to Q$ and $g : Q \to R$ be monotone functions. Prove that the function composition $g \circ f : P \to R$ is also monotone.

4. Find an order such that $F(X) = \{ f(x) \mid x \in X \}$ is a monotone function. Give a proof for your answer.
Exercise 2.3: Complete Lattices (9 = 3+3+3 Points)

Let \( | \) be the relation of divisibility: \( a | b \iff \exists t \in \mathbb{N} \setminus \{0\}, a \cdot t = b \). Which of the following are complete lattices? Justify your answers.

1. \((\mathbb{N}, |)\)
2. \((\mathbb{N} \setminus \{0\}, |)\)
3. \((\mathbb{N} \setminus \{0\} \cup \{\infty\}, |)\), where \( \forall t : t \cdot \infty = \infty \cdot t = \infty \)

Exercise 2.4: Complete Lattices (7 Points)

Which of the following Hasse diagrams represent complete lattices?

- (a)
- (b)
- (c)
- (d)
- (e)
- (f)
- (g)