Design and Analysis of Real-Time Systems

Foundations of Abstract Interpretation II

Jan Reineke

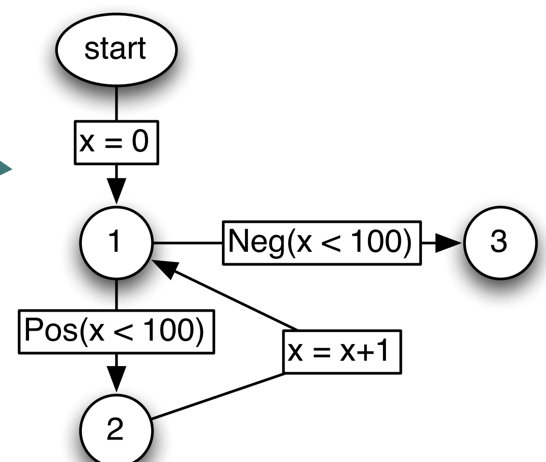
Advanced Lecture, Summer 2013

How to Compute the Least Fixed Point

Kleene Iteration:

$$\bot \leq f(\bot) \leq f^2(\bot) \leq f^3(\bot) \leq \ldots$$

Why is this increasing?
Will this reach the fixed point?
It will here:
But in general?



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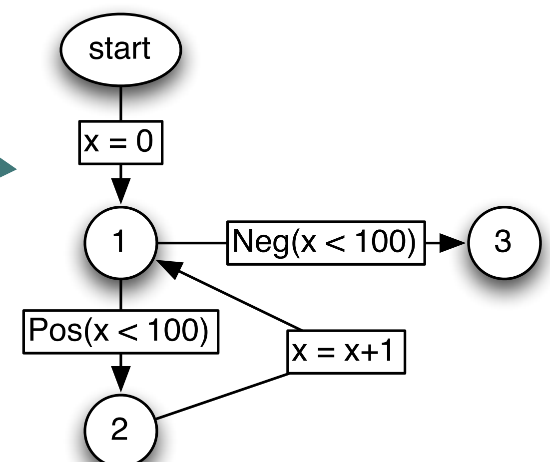

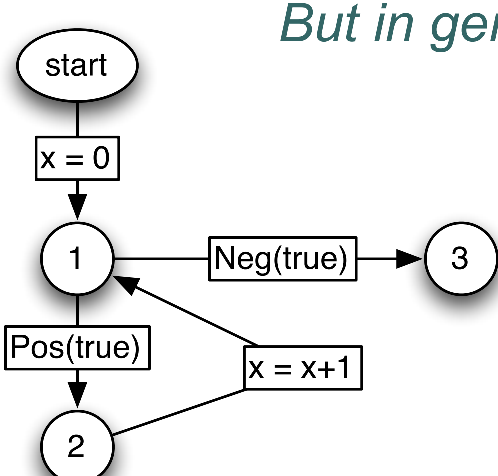

No!

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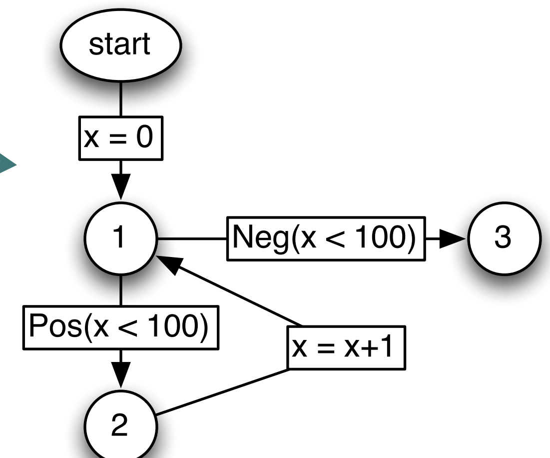
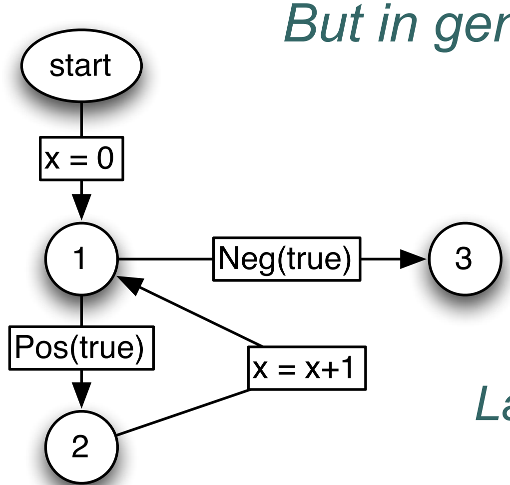
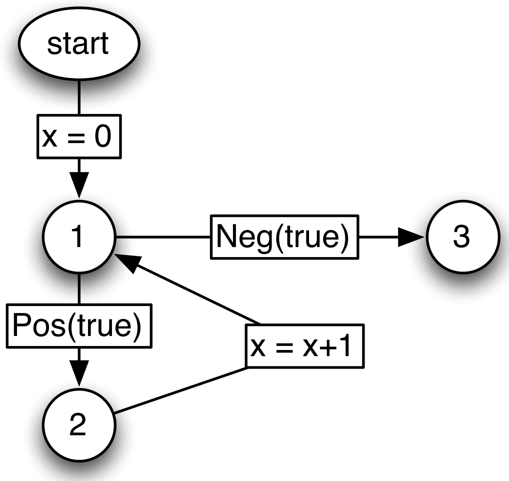




No!

Lattice has infinite ascending chains.

Infinite Ascending Chains



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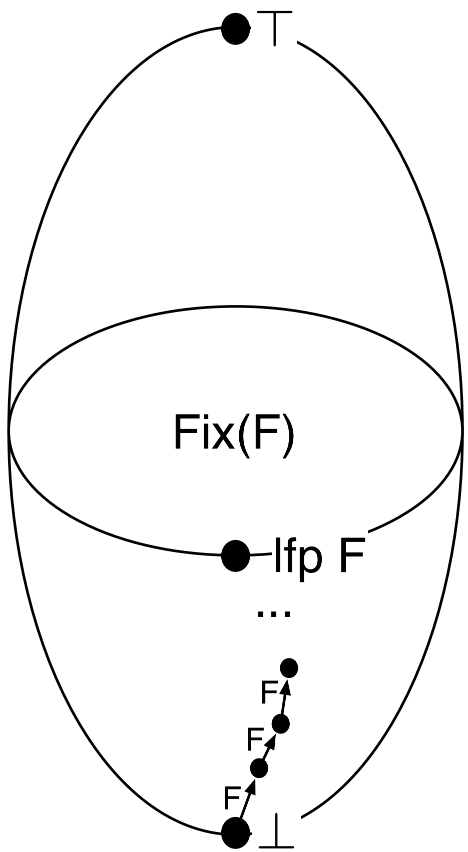
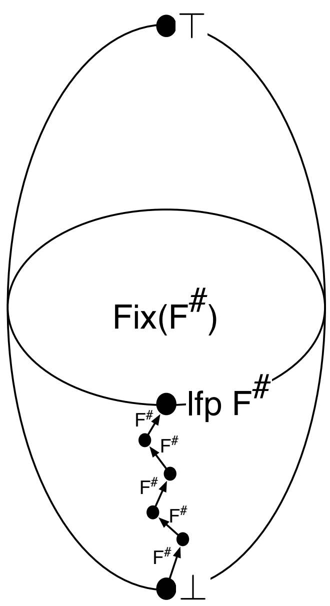


Think of an example of an infinite ascending chain.

Ascending Chain Condition

A partially-ordered set S satisfies the *ascending chain condition* if every strictly ascending sequence of elements is finite.



Theorem (Ascending Chain Condition):

Let $(S, \leq)$ *be a* complete lattice *set that satisfies the ascending chain condition, and let* $f : S \to S$ *be a monotone function. Then, there is an* $n \in \mathbb{N}$*, such that*

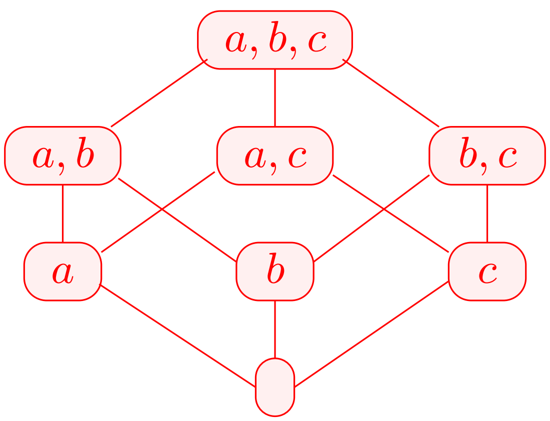
$$\mathit{lfp}\ f = f^n(\bot).$$

➔ *Length of longest ascending chain determines worst-case complexity of Kleene Iteration.*

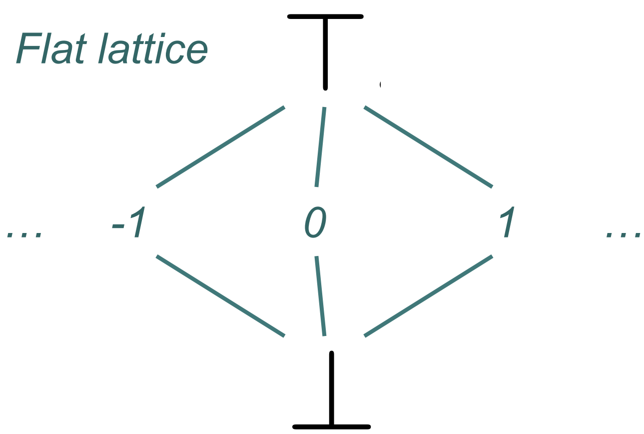
Ascending Chain Condition: Examples

A partially-ordered set S satisfies the *ascending chain condition* if every strictly ascending sequence of elements is finite.

Power set lattice



Flat lattice



→ *Ascending chain condition does not imply finite partially-ordered set!*

How about total function space lattice?
How about finite partially-ordered sets?

Recap: Abstract Interpretation

- Semantics-based approach to program analysis
- Framework to develop provably correct and terminating analyses

Ingredients:

- Concrete semantics: Formalizes meaning of a program ✓
- Abstract semantics
- Both semantics defined as fixpoints of monotone functions over some domain (✓)
- Relation between the two semantics establishing correctness

Abstract Semantics

Similar to concrete semantics:

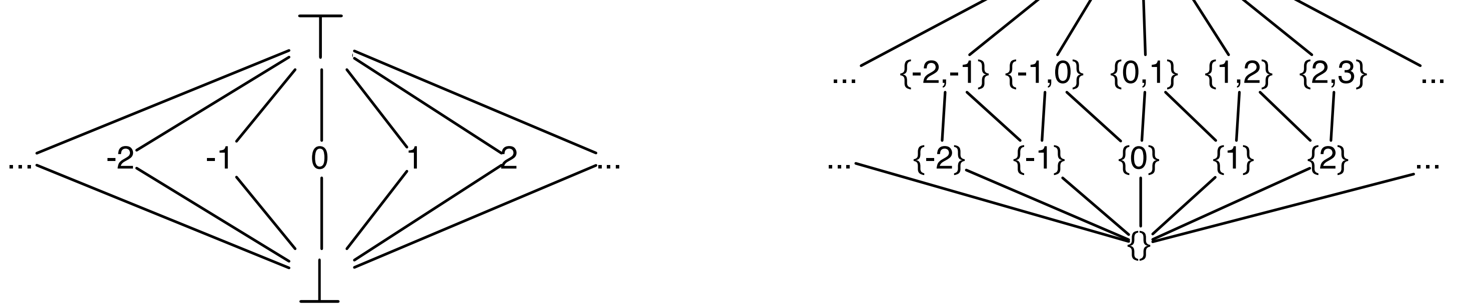
- A complete lattice $(L^{\#}, \leq)$ as the domain for abstract elements
- A monotone function $F^{\#}$ corresponding to the concrete function F
- Then the abstract semantics is the least fixed point of $F^{\#}$, lfp $F^{\#}$

If $F^{\#}$ "correctly approximates" F,

then lfp $F^{\#}$ "correctly approximates" lfp F.

An Example Abstract Domain for Values of Variables

$(\mathbb{Z}_{\bot}^{\top}, \leq)$ $(\mathcal{P}(\mathbb{Z}), \subseteq)$

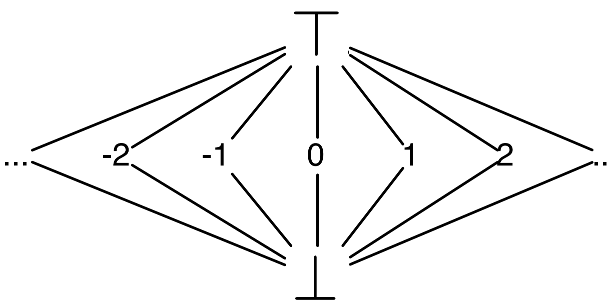


$$\gamma : \mathbb{Z}_{\bot}^{\top} \to \mathcal{P}(\mathbb{Z})$$

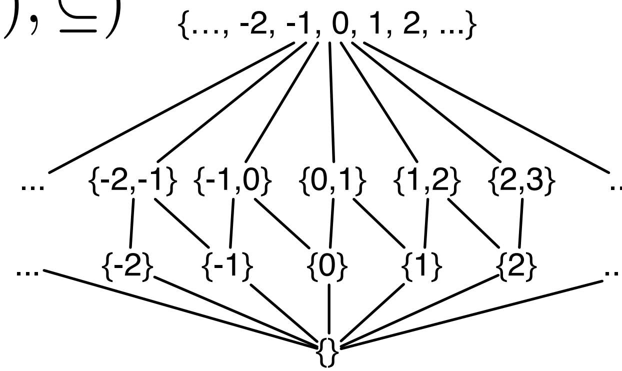
$$\alpha : \mathcal{P}(\mathbb{Z}) \to \mathbb{Z}_{\bot}^{\top}$$

An Example Abstract Domain for Values of Variables

$(\mathbb{Z}_\perp^\top, \leq)$



$(\mathcal{P}(\mathbb{Z}), \subseteq)$



How to relate the two?

➔ *Concretization function, specifying "meaning" of abstract values.*

$$\gamma : \mathbb{Z}_\perp^\top \rightarrow \mathcal{P}(\mathbb{Z})$$

➔ *Abstraction function: determines best representation concrete values.*

$$\alpha : \mathcal{P}(\mathbb{Z}) \rightarrow \mathbb{Z}_\perp^\top$$

Relation between the Abstract and Concrete Domains

$$\gamma(\top)$$

$$\gamma(\bot)$$

$$\gamma(x)$$

Relation between the Abstract and Concrete Domains

$$\gamma(\top) := \mathbb{Z}$$

$$\gamma(\bot) := \emptyset$$

$$\gamma(x) := \{x\}$$

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$$\alpha(A) := \begin{cases} \top \\ x \\ \bot \end{cases}$$

Relation between the Abstract and Concrete Domains

$$\gamma(\top) := \mathbb{Z}$$
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$$\alpha(A) := \begin{cases} \top & : |A| \geq 2 \\ x & : A = \{x\} \\ \bot & : A = \emptyset \end{cases}$$

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1. *Are these functions monotone?*
2. *Should they be?*
3. *What is the meaning of the partial order in the abstract domain?*
4. *What if we first abstract and the concretize?*

How to Compute in the Abstract Domain
Example: Multiplication on Flat Lattice

$*^{\#}$	$\top$	a	0	$\bot$
$\top$				
b				
0				
$\bot$				

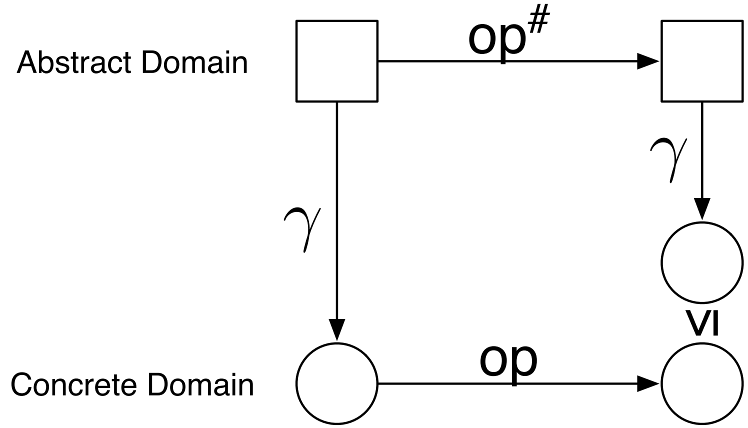
How to Compute in the Abstract Domain
Example: Multiplication on Flat Lattice

Denotes abstract version of operator

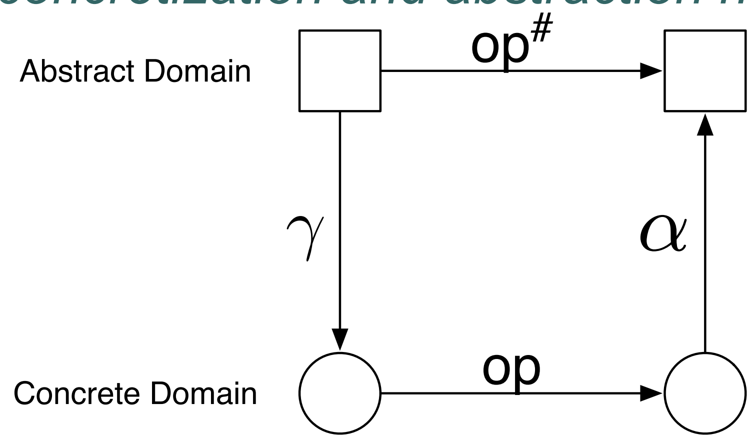
$*^{\#}$	$\top$	a	0	$\bot$
$\top$				
b				
0				
$\bot$				

How to Compute in the Abstract Domain: Correctness Conditions

Correctness Condition:



Correct by construction
(if concretization and abstraction have certain properties):



How to Compute in the Abstract Domain
Example: Multiplication on Flat Lattice

$$\top \quad 0 \quad \xrightarrow{*^{\#}}$$

How to Compute in the Abstract Domain
Example: Multiplication on Flat Lattice

$$
\begin{tikzcd}
\top \arrow[d, "\gamma"'] & 0 \arrow[d, "\gamma"'] \arrow[r, "*^{\#}"] & {} \\
\mathbb{Z} & \{0\} &
\end{tikzcd}
$$

How to Compute in the Abstract Domain
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How to Compute in the Abstract Domain
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$$
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\top \arrow[d, "\gamma"'] & 0 \arrow[d, "\gamma"'] \arrow[r, "*^{\#}"] & 0 \\
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\end{tikzcd}
$$

How to Compute in the Abstract Domain Example: Multiplication on Flat Lattice

$a \quad b \xrightarrow{\quad *^{\#} \quad}$

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Example: Multiplication on Flat Lattice

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\begin{tikzcd}
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\end{tikzcd}
$$

How to Compute in the Abstract Domain Example: Multiplication on Flat Lattice

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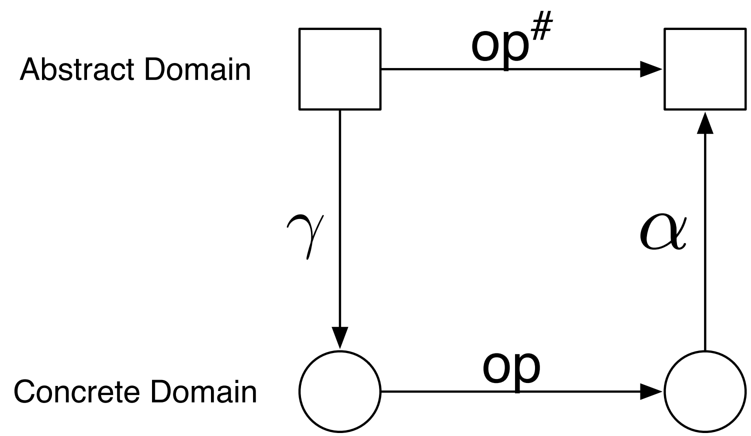
How to Compute in the Abstract Domain: Correct by Construction

Correct by construction
(if concretization and abstraction have certain properties):



"Certain properties": Notion of Galois connections:

Let $(L, \leq)$ *and* $(M, \sqsubseteq)$ *be partially ordered sets and* $\alpha \in L \to M$, $\gamma \in M \to L$. *We call* $(L, \leq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} (M, \sqsubseteq)$ *a* Galois connection *if* α *and* γ *are monotone functions and*

$$\begin{array}{rcl} l & \leq & \gamma(\alpha(l)) \\ \alpha(\gamma(m)) & \sqsubseteq & m \end{array}$$

for all $l \in L$ *and* $m \in M$.

Galois connections

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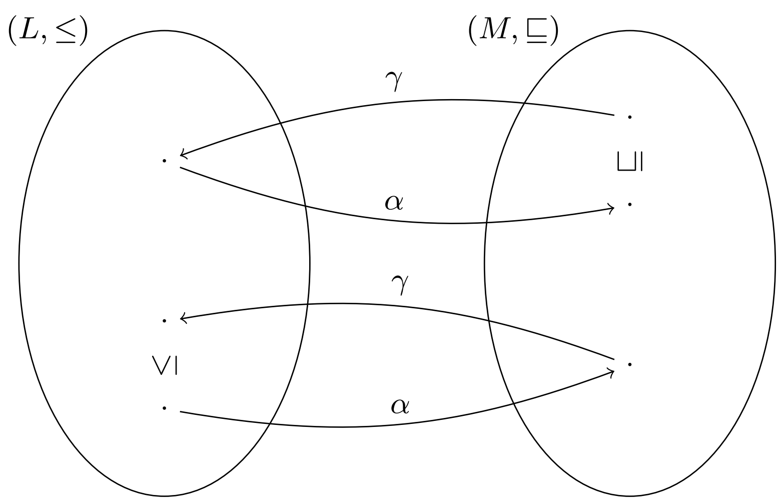
Graphically:



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Why monotone?

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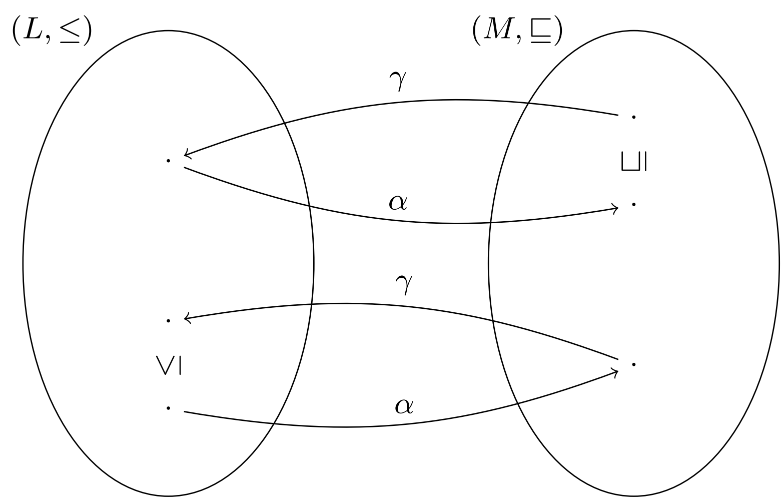

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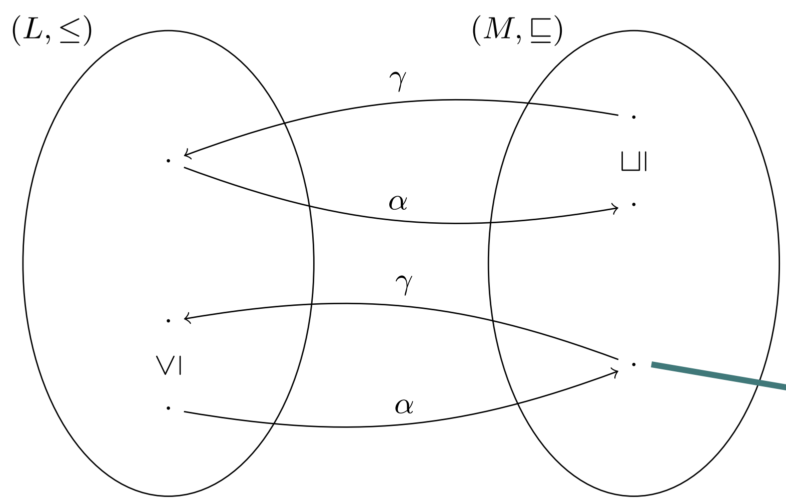

For soundness.

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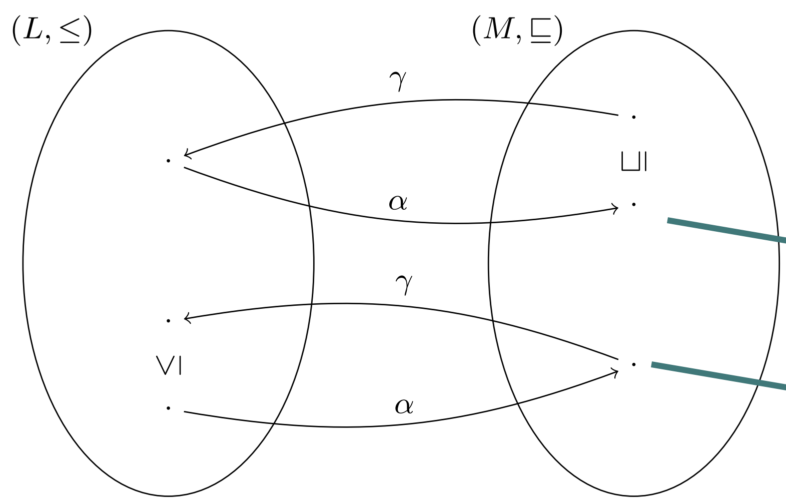
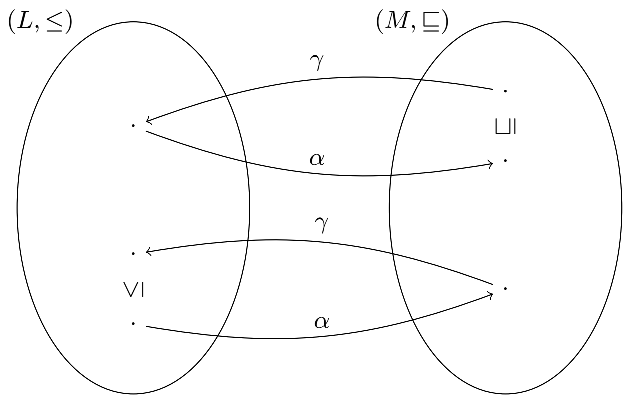


For precision.

For soundness.

Galois connections: Properties

Graphically:



Properties:

1) *Can be used to systematically construct correct (and in fact the most precise) abstract operations:* $op^{\#} = \alpha \circ op \circ \gamma$
2) *a) Abstraction function induces concretization function*
   b) Concretization function induces abstraction function

Galois connections: Properties

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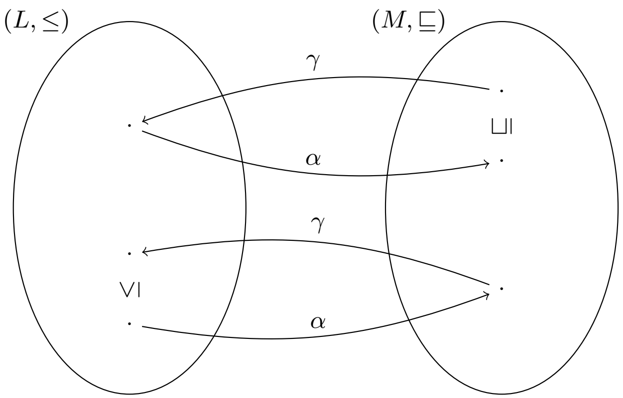

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Why?

How?

Abstracting Sets of Concrete States
Recap: Concrete States

Concrete states are not just sets of values...

Concrete states consist of variables and memory:

$$s = (\rho, \mu) \in States$$

$\rho : Vars \to int$ ← *Values of Variables*

$\mu : \mathbb{N} \to int$ ← *Contents of Memory*

$$States = (Vars \to int) \times (\mathbb{N} \to int)$$

$$States = (Vars \to \mathbb{Z}) \times (\mathbb{N} \to \mathbb{Z})$$

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$$\rho : Vars \to int$$

Values of Variables

$$\mu : \mathbb{N} \to int$$

Contents of Memory

$$\cancel{States = (Vars \to int) \times (\mathbb{N} \to int)}$$

$$States = (Vars \to \mathbb{Z}) \times (\mathbb{N} \to \mathbb{Z})$$

Abstracting Sets of Concrete States
Recap: Concrete States

Reachability semantics is defined on sets of states:

$$[\![statement]\!] \subseteq States \times States$$

$$[\![statement]\!] : \mathcal{P}(States) \to \mathcal{P}(States)$$

$$[\![statement]\!](S) := \{s' \mid \exists s \in S : (s, s') \in [\![statement]\!]\}$$

$$\mathcal{P}(States) = \mathcal{P}((Vars \to \mathbb{Z}) \times (\mathbb{N} \to \mathbb{Z}))$$

Relation between Concrete Domain and Abstract Domain

Concrete domain!

$$\mathcal{P}(States) = \\ \quad \mathcal{P}((Vars \to \mathbb{Z}) \times (\mathbb{N} \to \mathbb{Z}))$$

Abstract domain?

Relation between Concrete Domain and Abstract Domain

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$$\mathcal{P}(States) = \mathcal{P}((Vars \to \mathbb{Z}) \times (\mathbb{N} \to \mathbb{Z}))$$

Abstract domain?

$$\widehat{States} = Vars \to \mathbb{Z}_{\bot}^{\top}$$

Relation between Concrete Domain and Abstract Domain

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Abstract domain?

$$\widehat{States} = \; Vars \to \mathbb{Z}_{\bot}^{\top}$$

Relation between the two?

→ For ease of understanding, introduce Intermediate domain:

$$Power\widehat{SetS}tates = \; Vars \to \mathcal{P}(\mathbb{Z})$$

Relation between Concrete Domain and Intermediate Domain

Concrete domain:

$$\mathcal{P}(States) = \mathcal{P}((Vars \to \mathbb{Z}) \times (\mathbb{N} \to \mathbb{Z}))$$

Intermediate domain:

$$\widehat{PowerSetStates} = Vars \to \mathcal{P}(\mathbb{Z})$$

Abstraction:

Concretization:

Relation between Concrete Domain and Intermediate Domain

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Abstraction:

$$\alpha_{C,I} : \mathcal{P}(States) \to \widehat{PowerSetStates}$$

$$\alpha_{C,I}(C) := \lambda x \in Vars.\{v(x) \in \mathbb{Z} \mid (v,m) \in C\}$$

Concretization:

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Concretization:

$$\gamma_{I,C} : \widehat{PowerSetStates} \to \mathcal{P}(States)$$

$$\gamma_{I,C}(\widehat{c}) := \{(v,m) \in States \mid \forall x \in Vars : v(x) \in \widehat{c}(x)\}$$

Relation between Intermediate Domain and Abstract Domain

Intermediate domain:

$$\widehat{PowerSetStates} = Vars \to \mathcal{P}(\mathbb{Z})$$

Abstract domain:

$$\widehat{States} = Vars \to \mathbb{Z}_{\bot}^{\top}$$

Abstraction:

Concretization:

Relation between Intermediate Domain and Abstract Domain

Intermediate domain:

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Abstraction and Concretization functions from before!

Could plug in other abstractions for sets of values…

Relation between Concrete Domain and Abstract Domain

Concrete domain:

$$\mathcal{P}(States) = \mathcal{P}((Vars \to \mathbb{Z}) \times (\mathbb{N} \to \mathbb{Z}))$$

Abstract domain:

$$\widehat{States} = Vars \to \mathbb{Z}_{\perp}^{\top}$$

Abstraction:

$$\alpha_{C,A} : \mathcal{P}(States) \to \widehat{States}$$

$$\alpha_{C,A} := \alpha_{I,A} \circ \alpha_{C,I}$$

Concretization:

$$\gamma_{A,C} : \widehat{States} \to \mathcal{P}(States)$$

$$\gamma_{A,C} := \gamma_{I,C} \circ \gamma_{A,I}$$

Relation between Concrete Domain and Abstract Domain

Concrete domain:

$$\mathcal{P}(States) = \mathcal{P}((Vars \to \mathbb{Z}) \times (\mathbb{N} \to \mathbb{Z}))$$

Abstract domain:

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Concretization:

$$\gamma_{A,C} : \widehat{States} \to \mathcal{P}(States)$$

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Galois connections can be composed to obtain new Galois connections.

Meaning of Statements in the Abstract Domain

$$[\![R = e]\!]^{\#}(\widehat{a}) := \widehat{a}[R \mapsto [\![e]\!]^{\#}(\widehat{a})]$$

$$[\![R = M[e]]\!]^{\#}(\widehat{a}) := \widehat{a}[R \mapsto \top]$$

$$[\![M[e_1] = e_2]\!]^{\#}(\widehat{a}) := \widehat{a}$$

$$[\![Pos(e)]\!]^{\#}(\widehat{a}) := \widehat{a}$$

$$[\![Neg(e)]\!]^{\#}(\widehat{a}) := \widehat{a}$$

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Can this be done better?

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$$[\![R = M[e]]\!]^{\#}(\widehat{a}) := \widehat{a}[R \mapsto \top]$$

$$[\![M[e_1] = e_2]\!]^{\#}(\widehat{a}) := \widehat{a}$$

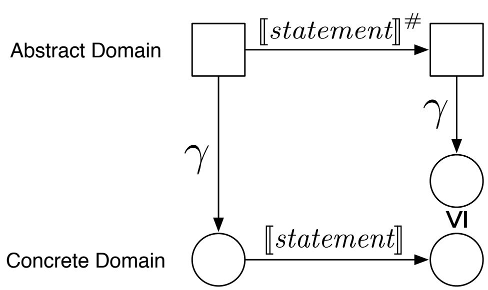
$$[\![Pos(e)]\!]^{\#}(\widehat{a}) := \widehat{a}$$

$$[\![Neg(e)]\!]^{\#}(\widehat{a}) := \widehat{a}$$

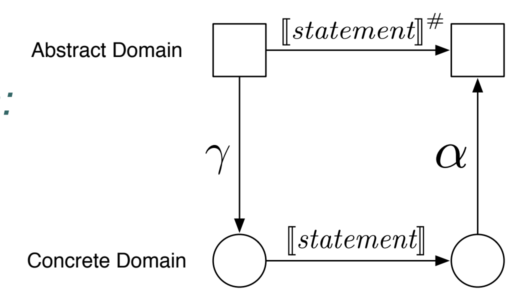
Can this be done better?

Again:

For Correctness:



For the best possible precision:



Meaning of Expressions

Evaluation of expressions is as expected:

$$[\![x]\!]^{\#}(\widehat{a}) := \widehat{a}(x) \qquad\qquad if\ x \in Vars$$

$$[\![e_1\ \ op\ e_2]\!]^{\#}(\widehat{a}) := [\![e_1]\!]^{\#}(\widehat{a})\ \ op^{\#}\ \ [\![e_2]\!]^{\#}(\widehat{a})$$

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As we have seen earlier!

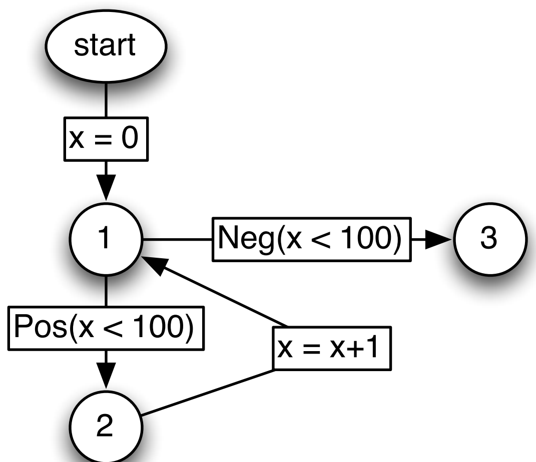
Putting it all together: The Abstract Reachability Semantics

Abstract Reachability Semantics captured as least fixed point of:

$$\widehat{Reach} : V \to \widehat{States}$$

$$\widehat{Reach}(start) = \top$$

$$\forall v' \in V \setminus \{start\} : \widehat{Reach}(v') = \bigsqcup_{v \in V, (v,v') \in E} [\![labeling(v, v')]\!]^{\#}(\widehat{Reach}(v))$$



$$\widehat{Reach}(1) = [\![labeling(start, 1)]\!]^{\#}(\widehat{Reach}(start)) \sqcup [\![labeling(2, 1)]\!](\widehat{Reach}(2))$$

$$\widehat{Reach}(2) = [\![labeling(1, 2)]\!]^{\#}(\widehat{Reach}(1))$$

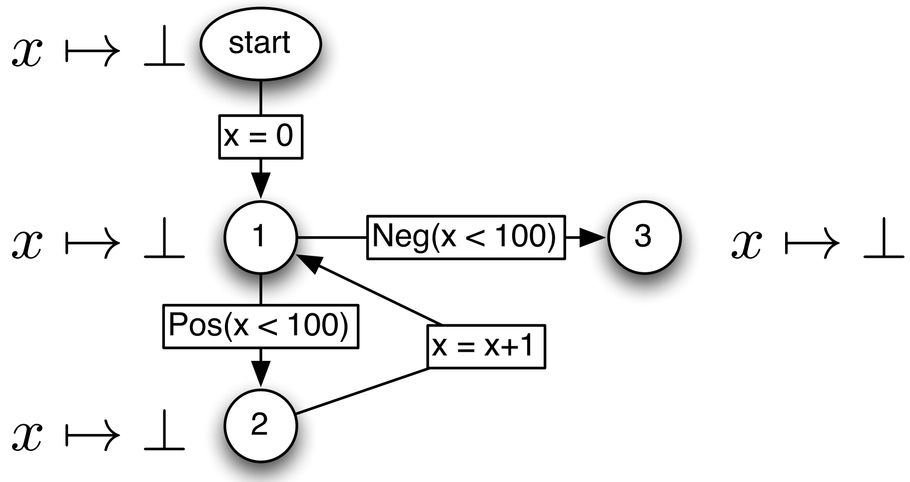
$$\widehat{Reach}(3) = [\![labeling(1, 3)]\!]^{\#}(\widehat{Reach}(1))$$

$$\widehat{Reach}(1) = [\![x = 0]\!]^{\#}(\widehat{Reach}(start)) \sqcup [\![x = x + 1]\!]^{\#}(\widehat{Reach}(2))$$

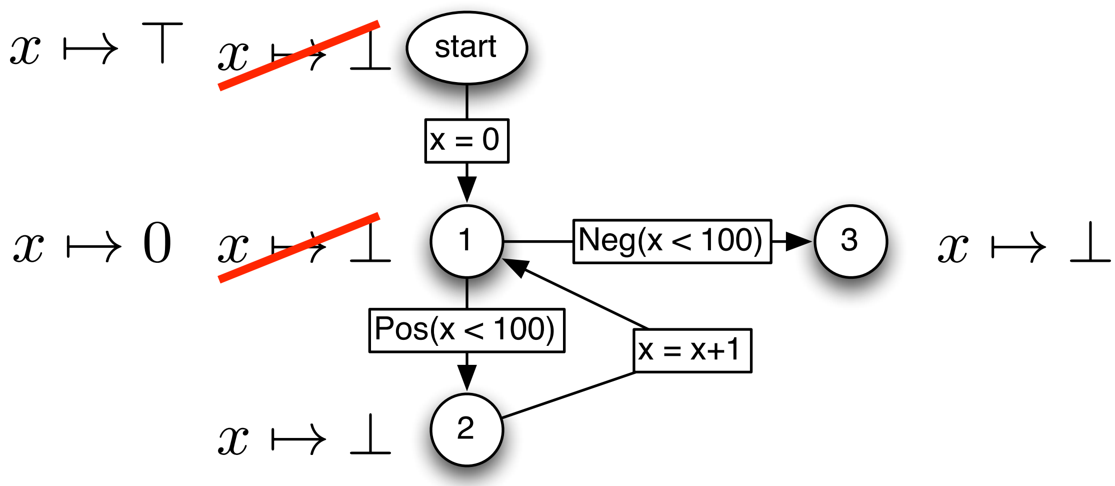
$$\widehat{Reach}(2) = [\![Pos(x < 100)]\!]^{\#}(\widehat{Reach}(1))$$

$$\widehat{Reach}(3) = [\![Neg(x < 100)]\!]^{\#}(\widehat{Reach}(1))$$

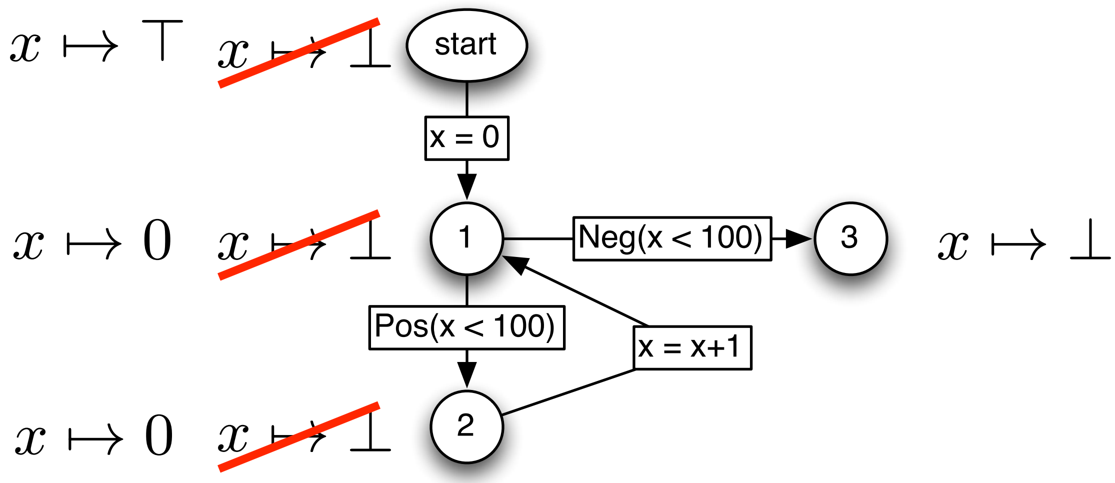
Example: Kleene Iteration to Compute Abstract Reachability Semantics



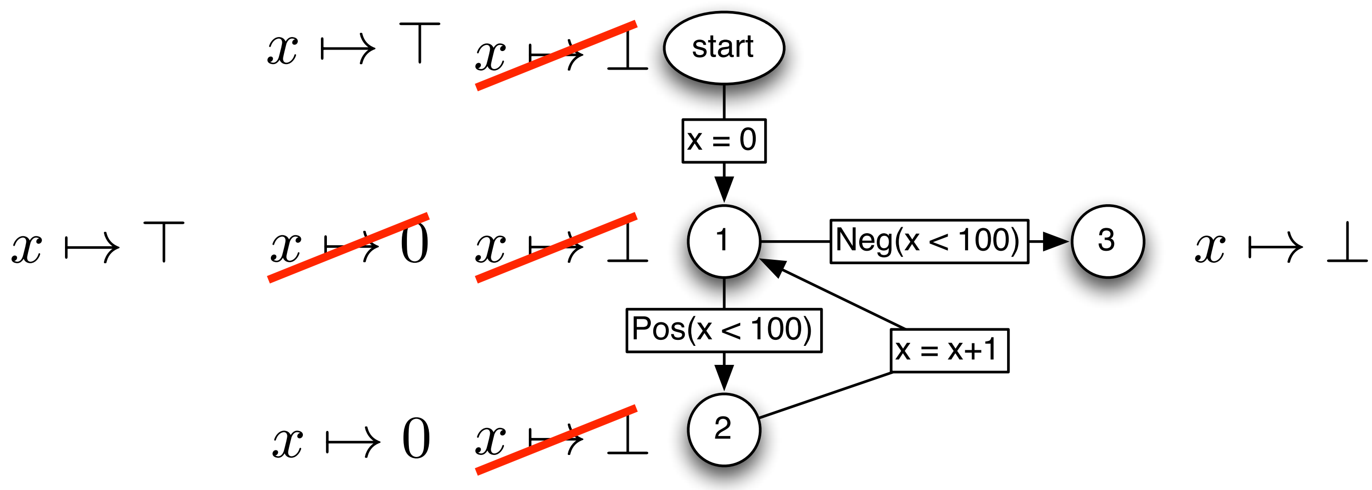
Example: Kleene Iteration to Compute Abstract Reachability Semantics



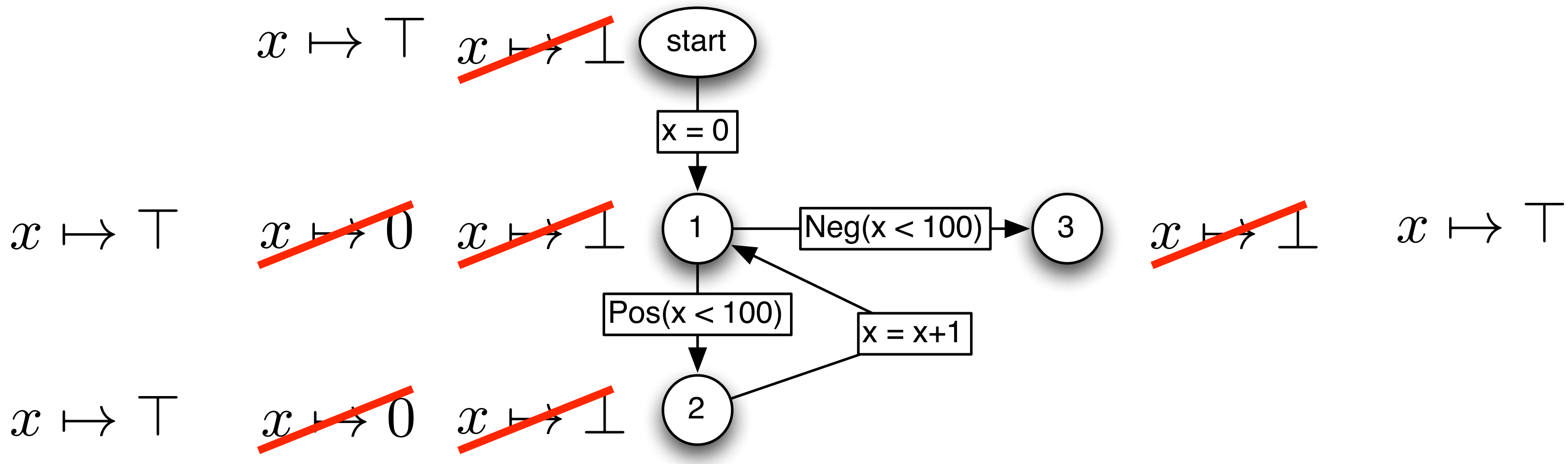
Example: Kleene Iteration to Compute Abstract Reachability Semantics



Example: Kleene Iteration to Compute Abstract Reachability Semantics

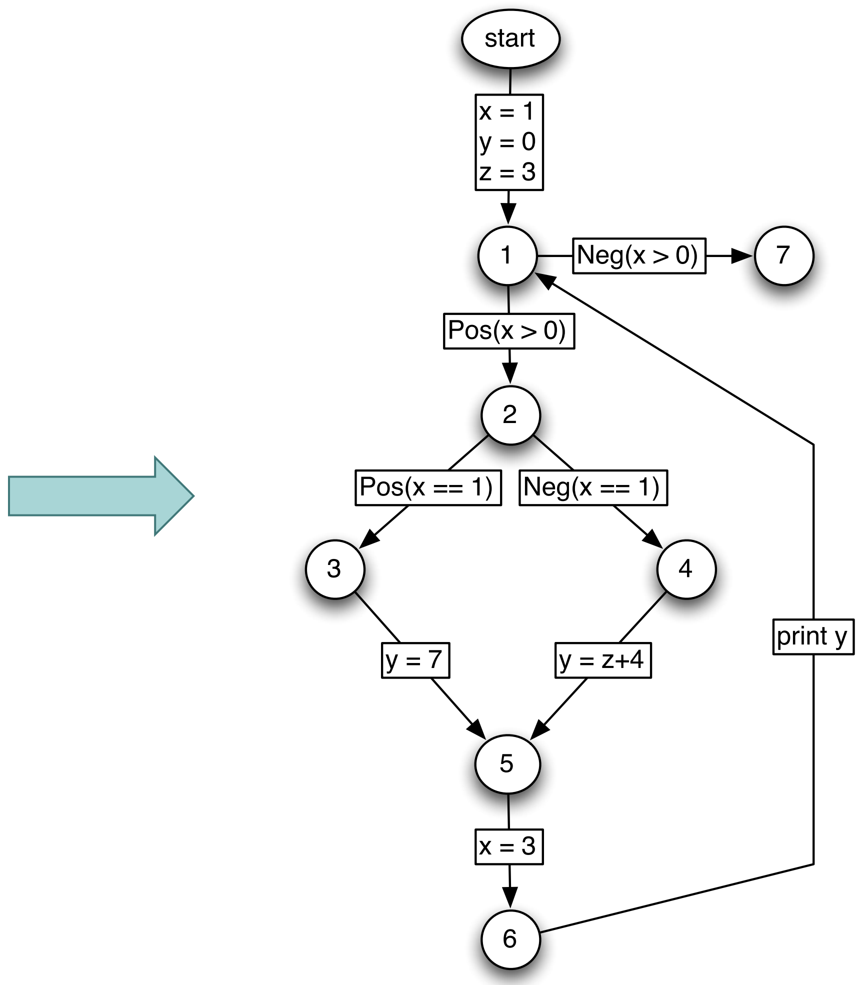


Example: Kleene Iteration to Compute Abstract Reachability Semantics



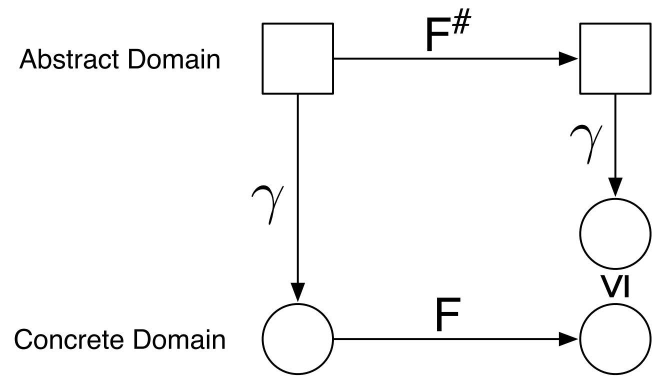
Example II: Kleene Iteration to Compute Abstract Reachability Semantics

```
y = 0;
x = 1;
z = 3;
while (x > 0) {
    if (x == 1) {
        y = 7;
    }
    else {
        y = z+4;
    }
    x = 3;
    print y;
}
```



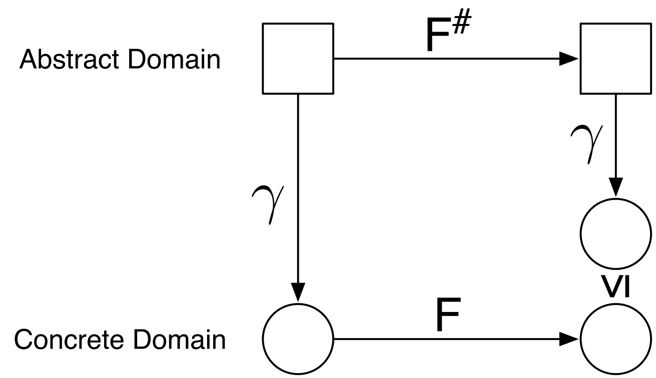
The Abstract Transformer $F^{\#}$

Local Correctness Condition:

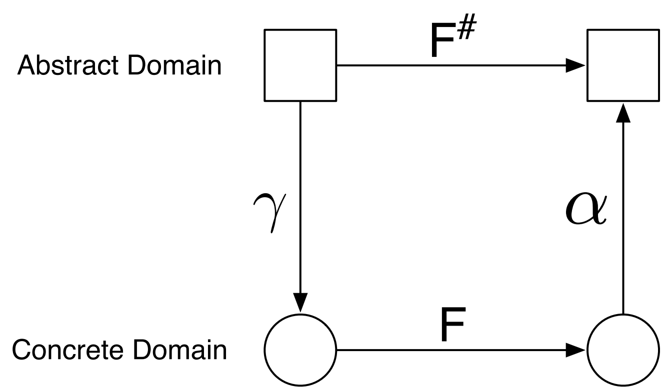


The Abstract Transformer $F^{\#}$

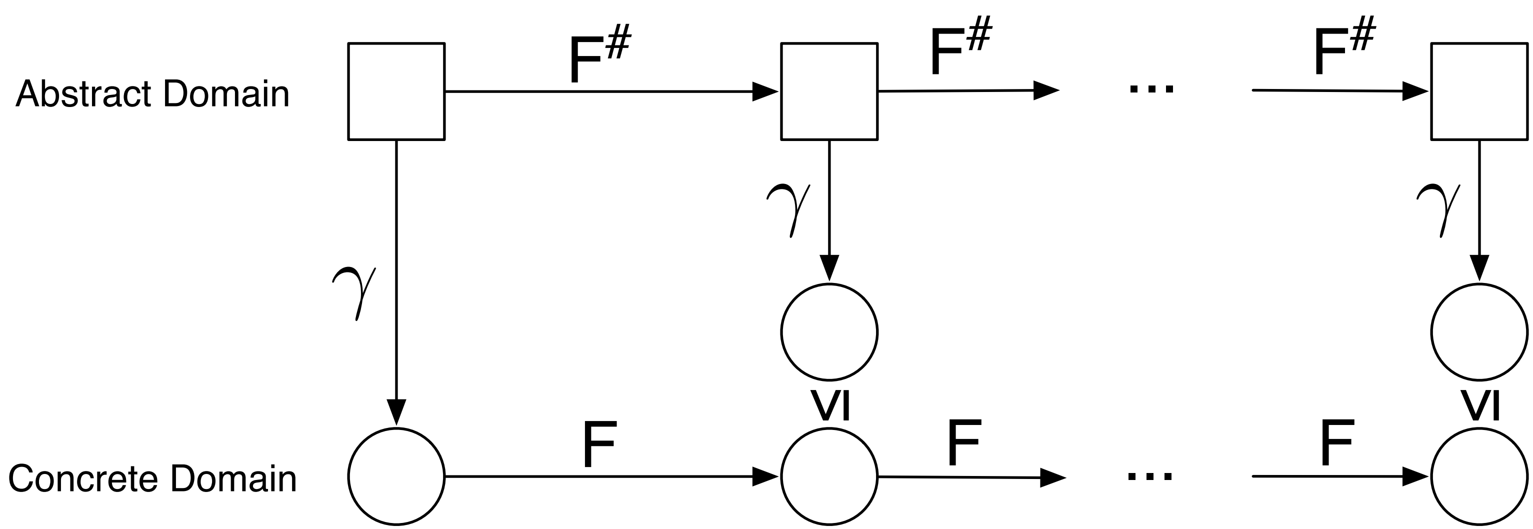
Local Correctness Condition:



Correct by construction
(if concretization and abstraction have certain properties):



From Local to Global Correctness: Kleene Iteration



Fixpoint Transfer Theorem

Let $(L, \leq)$ and $(L^\#, \leq^\#)$ be two lattices, $\gamma : L^\# \to L$ a monotone function, and $F : L \to L$ and $F^\# \to F^\#$ two monotone functions, with

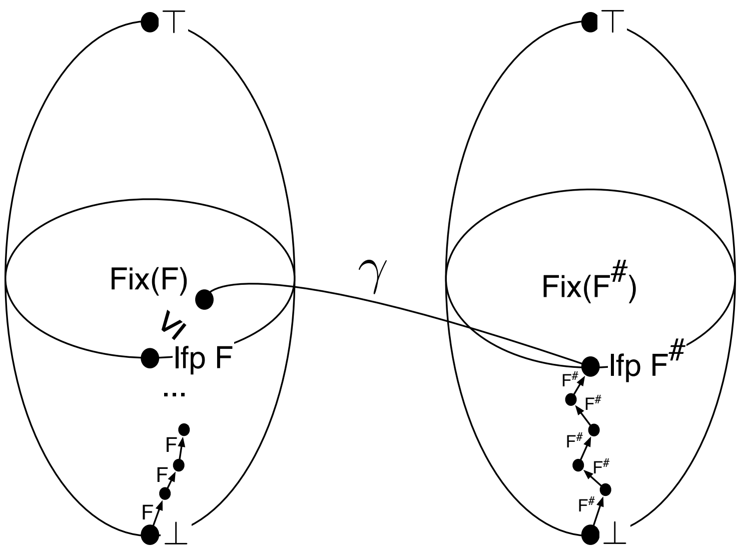
$$\forall l^\# \in L^\# : \gamma(F^\#(l^\#)) \geq F(\gamma(l^\#)).$$

Local Correctness

Then:

$$\mathit{lfp}\ F \leq \gamma(\mathit{lfp}\ F^\#).$$

Global Correctness



Outlook: Other Abstractions

- Signs
- Parity
- Intervals
- Octagons
- Congruence