Outline

1 Caches

2 Cache Analysis for Least-Recently-Used

3 Beyond Least-Recently-Used
   - Predictability Metrics
   - Relative Competitiveness
   - Sensitivity – Caches and Measurement-Based Timing Analysis

4 Summary
Outline

1. Caches

2. Cache Analysis for Least-Recently-Used

3. Beyond Least-Recently-Used
   - Predictability Metrics
   - Relative Competitiveness
   - Sensitivity – Caches and Measurement-Based Timing Analysis

4. Summary
Uncertainty in WCET Analysis

- Amount of uncertainty determines precision of WCET analysis
- Uncertainty in cache analysis depends on replacement policy

![Diagram showing BCET, ACET, WCET, upper bound, execution time, uncertainty, variation due to inputs, and penalty]
Uncertainty in Cache Analysis

1. Initial cache contents unknown.
2. Need to combine information.
3. Cannot resolve address of \( z \).
Uncertainty in Cache Analysis

1. Initial cache contents unknown.

- 1. Initial cache contents unknown.
Uncertainty in Cache Analysis

1. Initial cache contents unknown.
2. Need to combine information.
Uncertainty in Cache Analysis

1. Initial cache contents unknown.
2. Need to combine information.
3. Cannot resolve address of $z$.

Amount of uncertainty determined by ability to recover information.
Uncertainty in Cache Analysis

1. Initial cache contents unknown.
2. Need to combine information.
3. Cannot resolve address of $z$.

$\rightarrow$ Amount of uncertainty determined by ability to recover information
Predictability Metrics

Evict → Fill

Sequence: \( \langle a, \ldots, e, f, g, h \rangle \)
Meaning of Metrics

- **Evict**
  - Number of accesses to obtain *any* *may*-information.
  - I.e. when can an analysis predict any cache misses?

- **Fill**
  - Number of accesses to complete *may*- and *must*-information.
  - I.e. when can an analysis predict each access?

→ Evict and Fill bound the precision of *any* static cache analysis. Can thus serve as a benchmark for analyses.
Evaluation of Least-Recently-Used

- LRU “forgets” about past quickly:
  - cares about most-recent access to each block only
  - order of previous accesses irrelevant

In the example: Evict = Fill = 4

In general: \( \text{Evict}(k) = \text{Fill}(k) = k \), where \( k \) is the associativity of the cache
Evaluation of First-In First-Out (sketch)

- Like LRU in the miss-case
- But: “Ignores” hits

In the worst-case $k - 1$ hits and $k$ misses: $(k = \text{associativity})$

- Evict$(k) = 2k - 1$

Another $k$ accesses to obtain complete knowledge:

- Fill$(k) = 3k - 1$
Evaluation of Pseudo-LRU (sketch)

- Tree-bits point to block to be replaced

![Diagram showing tree bits and blocks](attachment:image.png)

- Accesses “rejuvenate” neighborhood
  - Active blocks keep their (inactive) neighborhood in the cache

- Analysis yields:
  - Evict\((k) = \frac{k}{2} \log_2 k + 1\)
  - Fill\((k) = \frac{k}{2} \log_2 k + k - 1\)
### Evaluation of Policies

<table>
<thead>
<tr>
<th>Policy</th>
<th>Evict((k))</th>
<th>Fill((k))</th>
<th>Evict((8))</th>
<th>Fill((8))</th>
</tr>
</thead>
<tbody>
<tr>
<td>LRU</td>
<td>(k)</td>
<td>(k)</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>FIFO</td>
<td>(2k - 1)</td>
<td>(3k - 1)</td>
<td>15</td>
<td>23</td>
</tr>
<tr>
<td>MRU</td>
<td>(2k - 2)</td>
<td>(\infty/3k - 4)</td>
<td>14</td>
<td>(\infty/20)</td>
</tr>
<tr>
<td>PLRU</td>
<td>(\frac{k}{2} \log_2 k + 1)</td>
<td>(\frac{k}{2} \log_2 k + k - 1)</td>
<td>13</td>
<td>19</td>
</tr>
</tbody>
</table>

- LRU is optimal w.r.t. metrics.
- Other policies are much less predictable.
- Use LRU if predictability is a concern.
- How to obtain *may*- and *must*-information within the given limits for other policies?
Outline

1. Caches
2. Cache Analysis for Least-Recently-Used
3. Beyond Least-Recently-Used
   - Predictability Metrics
   - Relative Competitiveness
   - Sensitivity – Caches and Measurement-Based Timing Analysis
4. Summary
Relative Competitiveness

- **Competitiveness** (Sleator and Tarjan, 1985): worst-case performance of an online policy *relative to the optimal offline policy*
  - used to evaluate online policies

- **Relative competitiveness** (Reineke and Grund, 2008): worst-case performance of an online policy *relative to another online policy*
  - used to derive local and global cache analyses
Definition – Relative Miss-Competitiveness

Notation

\[ m_P(p, s) = \text{number of misses that policy } P \text{ incurs on} \]
\[ \text{access sequence } s \in M^* \text{ starting in state } p \in C_P \]
Definition – Relative Miss-Competitiveness

Notation

\[ m_P(p, s) = \text{number of misses that policy } P \text{ incurs on} \]
\[ \text{access sequence } s \in M^* \text{ starting in state } p \in C^P \]

Definition (Relative miss competitiveness)

Policy \( P \) is \((k, c)\)-miss-competitive relative to policy \( Q \) if

\[ m_P(p, s) \leq k \cdot m_Q(q, s) + c \]

for all access sequences \( s \in M^* \) and cache-set states \( p \in C^P, q \in C^Q \) that are compatible \( p \sim q \).
Definition – Relative Miss-Competitiveness

Notation

\[ m_P(p, s) = \text{number of misses that policy } P \text{ incurs on access sequence } s \in M^* \text{ starting in state } p \in C_P \]

Definition (Relative miss competitiveness)

Policy \( P \) is \((k, c)\)-miss-competitive relative to policy \( Q \) if

\[ m_P(p, s) \leq k \cdot m_Q(q, s) + c \]

for all access sequences \( s \in M^* \) and cache-set states \( p \in C_P, q \in C_Q \) that are compatible \( p \sim q \).

Definition (Competitive miss ratio of \( P \) relative to \( Q \))

The smallest \( k \), s.t. \( P \) is \((k, c)\)-miss-competitive rel. to \( Q \) for some \( c \).
Example – Relative Miss-Competitiveness

\( P \) is \((3, 4)\)-miss-competitive relative to \( Q \).
If \( Q \) incurs \( x \) misses, then \( P \) incurs at most \( 3 \cdot x + 4 \) misses.
Example – Relative Miss-Competitiveness

\( P \) is \((3, 4)\)-miss-competitive relative to \( Q \).

If \( Q \) incurs \( x \) misses, then \( P \) incurs at most \( 3 \cdot x + 4 \) misses.

Best: \( P \) is \((1, 0)\)-miss-competitive relative to \( Q \).
Example – Relative Miss-Competitiveness

\( P \) is \((3, 4)\)-miss-competitive relative to \( Q \).
If \( Q \) incurs \( x \) misses, then \( P \) incurs at most \( 3 \cdot x + 4 \) misses.

**Best:** \( P \) is \((1, 0)\)-miss-competitive relative to \( Q \).

**Worst:** \( P \) is not-miss-competitive (or \( \infty \)-miss-competitive) relative to \( Q \).
Example – Relative Hit-Competitiveness

\( P \) is \((\frac{2}{3}, 3)\)-hit-competitive relative to \( Q \).

If \( Q \) has \( x \) hits, then \( P \) has at least \( \frac{2}{3} \cdot x - 3 \) hits.
Example – Relative Hit-Competitiveness

\( P \) is \( \left( \frac{2}{3}, 3 \right) \)-hit-competitive relative to \( Q \).

If \( Q \) has \( x \) hits, then \( P \) has at least \( \frac{2}{3} \cdot x - 3 \) hits.

**Best:** \( P \) is \( (1, 0) \)-hit-competitive relative to \( Q \).
Equivalent to \( (1, 0) \)-miss-competitiveness.
Example – Relative Hit-Competitiveness

\( P \) is \( (\frac{2}{3}, 3) \)-hit-competitive relative to \( Q \).
If \( Q \) has \( x \) hits, then \( P \) has at least \( \frac{2}{3} \cdot x - 3 \) hits.

Best: \( P \) is \( (1, 0) \)-hit-competitive relative to \( Q \).
Equivalent to \( (1, 0) \)-miss-competitiveness.

Worst: \( P \) is \( (0, 0) \)-hit-competitive relative to \( Q \).
Analogue to \( \infty \)-miss-competitiveness.
Local Guarantees: (1, 0)-Competitiveness

Let $\mathbf{P}$ be (1, 0)-competitive relative to $\mathbf{Q}$:

$$m_\mathbf{P}(p, s) \leq 1 \cdot m_\mathbf{Q}(q, s) + 0$$

$$\Leftrightarrow m_\mathbf{P}(p, s) \leq m_\mathbf{Q}(q, s)$$
Local Guarantees: (1, 0)-Competitiveness

Let $P$ be (1, 0)-competitive relative to $Q$:

$$m_P(p, s) \leq 1 \cdot m_Q(q, s) + 0$$

$$\iff m_P(p, s) \leq m_Q(q, s)$$

1. If $Q$ “hits”, so does $P$, and
2. if $P$ “misses”, so does $Q$. 
Local Guarantees: \((1, 0)\)-Competitiveness

Let \(P\) be \((1, 0)\)-competitive relative to \(Q\):

\[
\begin{align*}
    m_P(p, s) &\leq 1 \cdot m_Q(q, s) + 0 \\
    \iff m_P(p, s) &\leq m_Q(q, s)
\end{align*}
\]

1. If \(Q\) “hits”, so does \(P\), and
2. if \(P\) “misses”, so does \(Q\).

As a consequence,

1. a \textit{must}-analysis for \(Q\) is also a \textit{must}-analysis for \(P\), and
2. a \textit{may}-analysis for \(P\) is also a \textit{may}-analysis for \(Q\).
Global Guarantees: \((k, c)\)-Competitiveness

Given: Global guarantees for policy \(Q\).

Wanted: Global guarantees for policy \(P\).
Global Guarantees: \((k, c)\)-Competitiveness

Given: Global guarantees for policy \(Q\).
Wanted: Global guarantees for policy \(P\).

1. Determine competitiveness of policy \(P\) relative to policy \(Q\).

\[ m_P \leq k \cdot m_Q + c \]
Global Guarantees: \((k, c)\)-Competitiveness

Given: Global guarantees for policy \(Q\).
Wanted: Global guarantees for policy \(P\).

1. Determine competitiveness of policy \(P\) relative to policy \(Q\).
   \[ m_P \leq k \cdot m_Q + c \]

2. Compute global guarantee for task \(T\) under policy \(Q\).
   \[ m_Q(T) \]
Global Guarantees: \((k, c)\)-Competitiveness

Given: Global guarantees for policy \(Q\).
Wanted: Global guarantees for policy \(P\).

1. Determine competitiveness of policy \(P\) relative to policy \(Q\).
   \[ m_P \leq k \cdot m_Q + c \]

2. Compute global guarantee for task \(T\) under policy \(Q\).
   \[ m_Q(T) \]

3. Calculate global guarantee on the number of misses for \(P\) using the global guarantee for \(Q\) and the competitiveness results of \(P\) relative to \(Q\).
   \[ m_P \leq k \cdot m_Q + c \quad m_Q(T) = m_P(T) \]
Relative Competitiveness: Automatic Computation

P and Q (here: FIFO and LRU) induce transition system:

\[
\begin{align*}
[h, h] & \xrightarrow{e} [eabc]_{\text{FIFO}}, [eabc]_{\text{LRU}} \\
[h, h] & \xrightarrow{c} [eabc]_{\text{FIFO}}, [ceab]_{\text{LRU}} \\
[m, m] & \xrightarrow{d} [abcd]_{\text{FIFO}}, [abcd]_{\text{LRU}} \\
[h, h] & \xrightarrow{a} [abcd]_{\text{FIFO}}, [abcd]_{\text{LRU}} \\
[m, m] & \xrightarrow{d} [abcd]_{\text{FIFO}}, [dabc]_{\text{LRU}} \\
[m, h] & \xrightarrow{d} [deab]_{\text{FIFO}}, [deab]_{\text{LRU}}
\end{align*}
\]

Legend
- [abcd]_{\text{FIFO}} Cache-set state
- (h, m) Memory access
- Misses in pairs of cache-set states

Competitive miss ratio = maximum ratio of misses in policy P to misses in policy Q in transition system
Problem: The induced transition system is $\infty$ large.

Observation: Only the relative positions of elements matter:

$$\begin{align*}
[abc]_{\text{LRU}}, [bde]_{\text{FIFO}} & \approx [fgl]_{\text{LRU}}, [ghm]_{\text{FIFO}} \\
\downarrow c \quad (h, m) & \quad \downarrow l \quad (h, m) \\
[cab]_{\text{LRU}}, [cbd]_{\text{FIFO}} & \approx [lfg]_{\text{LRU}}, [lgh]_{\text{FIFO}}
\end{align*}$$

Solution: Construct finite quotient transition system.
〜-Equivalent States in Running Example
Finite Quotient Transition System

Merging $\approx$-equivalent states yields a finite quotient transition system:
Competitive Ratio = Maximum Cycle Ratio

Competitive miss ratio =
maximum ratio of misses in policy $P$ to misses in policy $Q$
Competitive Ratio = Maximum Cycle Ratio

Competitive miss ratio =
maximum ratio of misses in policy $P$ to misses in policy $Q$

Maximum cycle ratio $= \frac{0+1+1}{0+1+0} = 2$
Tool Implementation

- Implemented in Java, called Relacs
- Interface for replacement policies
- Fully automatic
- Provides example sequences for competitive ratio and constant
- Analysis usually practically feasible up to associativity 8
  - limited by memory consumption
  - depends on similarity of replacement policies

Online version:
http://rw4.cs.uni-sb.de/~reineke/relacs
Generalizations

Identified patterns and proved generalizations by hand. Aided by example sequences generated by tool.
Generalizations

Identified patterns and proved generalizations by hand. Aided by example sequences generated by tool.

Previously unknown facts:

\[ \text{PLRU}(k) \text{ is } (1, 0) \text{ comp. rel. to } \text{LRU}(1 + \log_2 k), \]

\[ \rightarrow \text{LRU-must-analysis can be used for PLRU} \]
Generalizations

Identified patterns and proved generalizations by hand. Aided by example sequences generated by tool.

Previously unknown facts:

\[ \text{PLRU}(k) \text{ is } (1, 0) \text{ comp. rel. to } \text{LRU}(1 + \log_2 k), \]
\[ \rightarrow \text{LRU-}must\text{-analysis can be used for PLRU} \]

\[ \text{FIFO}(k) \text{ is } \left( \frac{1}{2}, \frac{k-1}{2} \right) \text{ hit-comp. rel. to } \text{LRU}(k), \text{ whereas} \]
\[ \text{LRU}(k) \text{ is } (0, 0) \text{ hit-comp. rel. to FIFO}(k), \text{ but} \]
Generalizations

Identified patterns and proved generalizations by hand.
Aided by example sequences generated by tool.

Previously unknown facts:

\[
\begin{align*}
\text{PLRU}(k) & \text{ is } (1, 0) \text{ comp. rel. to } \text{LRU}(1 + \log_2 k), \\
\rightarrow & \text{ LRU-} \textit{must}-\text{analysis can be used for PLRU} \\
\text{FIFO}(k) & \text{ is } \left(\frac{1}{2}, \frac{k-1}{2}\right) \text{ hit-comp. rel. to } \text{LRU}(k), \text{ whereas} \\
\text{LRU}(k) & \text{ is } (0, 0) \text{ hit-comp. rel. to } \text{FIFO}(k), \text{ but} \\
\text{LRU}(2k - 1) & \text{ is } (1, 0) \text{ comp. rel. to } \text{FIFO}(k), \text{ and} \\
\text{LRU}(2k - 2) & \text{ is } (1, 0) \text{ comp. rel. to } \text{MRU}(k). \\
\rightarrow & \text{ LRU-} \textit{may}-\text{analysis can be used for FIFO and MRU} \\
\rightarrow & \text{ optimal with respect to predictability metric Evict}
\end{align*}
\]
Generalizations

Identified patterns and proved generalizations by hand. Aided by example sequences generated by tool.

Previously unknown facts:

\[
\begin{align*}
\text{PLRU}(k) \text{ is } (1, 0) \text{ comp. rel. to } \text{LRU}(1 + \log_2 k), \\
\quad \rightarrow \text{LRU-}\textit{must}-\text{analysis can be used for PLRU} \\
\text{FIFO}(k) \text{ is } \left(\frac{1}{2}, \frac{k-1}{2}\right) \text{ hit-comp. rel. to } \text{LRU}(k), \text{ whereas} \\
\text{LRU}(k) \text{ is } (0, 0) \text{ hit-comp. rel. to } \text{FIFO}(k), \text{ but} \\
\text{LRU}(2k - 1) \text{ is } (1, 0) \text{ comp. rel. to } \text{FIFO}(k), \text{ and} \\
\text{LRU}(2k - 2) \text{ is } (1, 0) \text{ comp. rel. to } \text{MRU}(k). \\
\quad \rightarrow \text{LRU-}\textit{may}-\text{analysis can be used for FIFO and MRU} \\
\quad \rightarrow \text{optimal with respect to predictability metric Evict} \\
\end{align*}
\]

**FIFO-\textit{may}-analysis** used in the analysis of the branch target buffer of the MOTOROLA POWERPC 56x.
Outline

1. Caches

2. Cache Analysis for Least-Recently-Used

3. Beyond Least-Recently-Used
   - Predictability Metrics
   - Relative Competitiveness
   - Sensitivity – Caches and Measurement-Based Timing Analysis

4. Summary
Measurement-Based Timing Analysis

- Run program on a number of inputs and initial states.
- Combine measurements for basic blocks to obtain WCET estimation.
- Sensitivity Analysis demonstrates this approach may be dramatically wrong.
Run program on a number of inputs and initial states.

Combine measurements for basic blocks to obtain WCET estimation.

Sensitivity Analysis demonstrates this approach may be dramatically wrong.
Influence of Initial Cache State

variation due to initial cache state

BCET \quad \text{WCET} \quad \text{upper bound} \quad \text{execution time}

Definition (Miss sensitivity)

Policy $\mathcal{P}$ is $(k, c)$-miss-sensitive if

$$m_\mathcal{P}(p, s) \leq k \cdot m_\mathcal{P}(p', s) + c$$

for all access sequences $s \in M^*$ and cache-set states $p, p' \in C^\mathcal{P}$.
## Sensitivity Results

<table>
<thead>
<tr>
<th>Policy</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>LRU</td>
<td>1,2</td>
<td>1,3</td>
<td>1,4</td>
<td>1,5</td>
<td>1,6</td>
<td>1,7</td>
<td>1,8</td>
</tr>
<tr>
<td>FIFO</td>
<td>2,2</td>
<td>3,3</td>
<td>4,4</td>
<td>5,5</td>
<td>6,6</td>
<td>7,7</td>
<td>8,8</td>
</tr>
<tr>
<td>PLRU</td>
<td>1,2</td>
<td>—</td>
<td>∞</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>∞</td>
</tr>
<tr>
<td>MRU</td>
<td>1,2</td>
<td>3,4</td>
<td>5,6</td>
<td>7,8</td>
<td>MEM</td>
<td>MEM</td>
<td>MEM</td>
</tr>
</tbody>
</table>

- LRU is optimal. Performance varies in the least possible way.
- For FIFO, PLRU, and MRU the number of misses may vary strongly.
- Case study based on simple model of execution time by Hennessy and Patterson (2003): WCET may be 3 times higher than a measured execution time for 4-way FIFO.
Outline

1 Caches

2 Cache Analysis for Least-Recently-Used

3 Beyond Least-Recently-Used
   - Predictability Metrics
   - Relative Competitiveness
   - Sensitivity – Caches and Measurement-Based Timing Analysis

4 Summary
Summary

Cache Analysis for Least-Recently-Used

... efficiently represents sets of cache states by bounding the age of memory blocks from above and below.

... requires context-sensitivity for precision.
Summary

Cache Analysis for Least-Recently-Used

...efficiently represents sets of cache states by bounding the age of memory blocks from above and below.

...requires context-sensitivity for precision.

Predictability Metrics

...quantify the predictability of replacement policies.

 LRU is the most predictable policy.
Summary

Cache Analysis for Least-Recently-Used

...efficiently represents sets of cache states by bounding the age of memory blocks from above and below.

...requires context-sensitivity for precision.

Predictability Metrics

...quantify the predictability of replacement policies.

→ LRU is the most predictable policy.

Relative Competitiveness

...allows to derive guarantees on cache performance,

...yields first *may*-analyses for FIFO and MRU.
Summary

Cache Analysis for Least-Recently-Used

...efficiently represents sets of cache states by bounding the age of memory blocks from above and below.

...requires context-sensitivity for precision.

Predictability Metrics

...quantify the predictability of replacement policies.

\[ \rightarrow \text{LRU is the most predictable policy.} \]

Relative Competitiveness

...allows to derive guarantees on cache performance,

...yields first \textit{may}-analyses for FIFO and MRU.

Sensitivity Analysis

...determines the influence of initial state on cache performance.
Summary

Cache Analysis for Least-Recently-Used

...efficiently represents sets of cache states by bounding the age of memory blocks from above and below.

...requires context-sensitivity for precision.

Predictability Metrics

...quantify the predictability of replacement policies.

→ LRU is the most predictable policy.

Relative Competitiveness

...allows to derive guarantees on cache performance,

...yields first may-analyses for FIFO and MRU.

Sensitivity Analysis

...determines the influence of initial state on cache performance.

Thank you for your attention!
Summary

Cache Analysis for Least-Recently-Used

... efficiently represents sets of cache states by bounding the age of memory blocks from above and below.
... requires context-sensitivity for precision.

Predictability Metrics

... quantify the predictability of replacement policies.
→ LRU is the most predictable policy.

Relative Competitiveness

... allows to derive guarantees on cache performance,
... yields first may-analyses for FIFO and MRU.

Sensitivity Analysis

... determines the influence of initial state on cache performance.

Thank you for your attention!
Most-Recently-Used – MRU

MRU-bits record whether line was recently used

1. **[abcd]_{0101}**: b, d
2. **[ebcd]_{1101}**: e, b, d
3. **[ebcd]_{0010}**: c

→ Never converges
Pseudo-LRU – PLRU

Initial cache-set state $[a, b, c, d]_{110}$.

After a miss on e. State: $[a, b, e, d]_{011}$.

After a hit on a. State: $[a, b, e, d]_{111}$.

After a miss on f. State: $[a, b, e, f]_{010}$.

Hit on a “rejuvenates” neighborhood; “saves” b from eviction.
May- and Must-Information

\[
\text{May}^P(s) := \bigcup_{p \in C^P} \text{CC}_P(\text{update}_P(p, s))
\]

\[
\text{Must}^P(s) := \bigcap_{p \in C^P} \text{CC}_P(\text{update}_P(p, s))
\]

\[
\text{may}^P(n) := \left| \text{May}^P(s) \right|, \text{where } s \in S^\neq \subsetneq M^*, |s| = n
\]

\[
\text{must}^P(n) := \left| \text{Must}^P(s) \right|, \text{where } s \in S^\neq \subsetneq M^*, |s| = n
\]

\( S^\neq \): set of finite access sequences with pairwise different accesses
Definitions of Metrics

\[
\text{Evict}^P := \min \left\{ n \mid \text{may}^P(n) \leq n \right\},
\]

\[
\text{Fill}^P := \min \left\{ n \mid \text{must}^P(n) = k \right\},
\]

where \( k \) is \( P \)'s associativity.
Let $P(k)$ be $(1, 0)$-miss-competitive relative to policy $Q(l)$, then

(i) $Evict^P(k) \geq Evict^Q(l)$,

(ii) $mls^P(k) \geq mls^Q(l)$. 
Alternative Pred. Metrics ↔ Rel. Competitiveness

Let $l$ be the smallest associativity, such that $\text{LRU}(l)$ is $(1, 0)$-miss-competitive relative to $P(k)$. Then

$$\text{Alt-Evict}^P(k) = l.$$ 

Let $l$ be the greatest associativity, such that $P(k)$ is $(1, 0)$-miss-competitive relative to $\text{LRU}(l)$. Then

$$\text{Alt-mls}^P(k) = l.$$
Size of Transition System

\[ 2^{l+l'} \cdot \sum_{i=0}^{k} \binom{k}{i} \cdot \sum_{i'=0}^{k'} \binom{k'}{i'} \cdot \sum_{j=0}^{\min\{i,i'\}} \binom{j}{i} \binom{j}{i'} j! \]

status bits of P and Q
non-empty lines in P
non-empty lines in Q
number of overlappings in non-empty lines

\[ \sum_{j=0}^{\min\{k,k'\}} \binom{k}{j} \binom{k'}{j} j! \leq k! \cdot k'! \sum_{j=0}^{\infty} \frac{1}{(k-j)!j!(k'-j)!} \]

\[ \leq k! \cdot k'! \sum_{j=0}^{\infty} \frac{1}{j!} = e \cdot k! \cdot k'! \]

This can be bounded by

\[ 2^{l+l'+k+k'} \leq \|C_k^l \times C_k^{l'}\| \approx 2^{l+l'+k+k'} \cdot e \cdot k! \cdot k'! \]

bound on number of overlappings
Compatible States

\[ i^P = [\bot \bot \bot \bot]_P \approx i^Q = [\bot \bot \bot \bot]_Q \]

\[ update_P(i^P, s) \approx update_Q(i^Q, s) \]

\[ p \approx q \]
Let $\mathbf{P}$ be $(1, 0)$-competitive relative to $\mathbf{Q}$, then

\[
m_{\mathbf{P}}(p, \langle x \rangle) = 1 \quad \Rightarrow \quad m_{\mathbf{Q}}(q, \langle x \rangle) = 1
\]
(1, 0)-Competitiveness and May/Must-Analyses

∀p ∈ P : m_P(p, ⟨x⟩) = 1

∀q ∈ Q : m_Q(q, ⟨x⟩) = 1
Case Study: Impact of Sensitivity

- Simple model of execution time from Hennessy & Patterson (2003)
- \( \text{CPI}_{hit} \) = Cycles per instruction assuming cache hits only
- \( \frac{\text{Memory accesses}}{\text{Instruction}} \) including instruction and data fetches

\[
\frac{T_{wc}}{T_{meas}} = \frac{\text{CPI}_{hit} + \frac{\text{Memory accesses}}{\text{Instruction}} \times \text{Miss rate}_{wc} \times \text{Miss penalty}}{\text{CPI}_{hit} + \frac{\text{Memory accesses}}{\text{Instruction}} \times \text{Miss rate}_{meas} \times \text{Miss penalty}}
\]

\[
= \frac{1.5 + 1.2 \times 0.20 \times 50}{1.5 + 1.2 \times 0.05 \times 50} = \frac{13.5}{4.5} = 3
\]