# Design and Analysis of Real-Time Systems SS 2013

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*Deadline: Thursday, May 16, 2013, 14:15*

# Assignment 3

## Problem 1: Galois Connections (7 Points)

Which of the following examples are not Galois connections? Find counterexamples for those cases!

1. $(\mathbb{R}, \leq) \overset{\gamma}{\underset{\alpha}{\leftrightarrows}} (\mathbb{R}, \leq)$, with $\alpha(x) = \lceil x \rceil$ and $\gamma(y) = \lfloor y \rfloor$
2. $(\mathbb{R}, \leq) \overset{\gamma}{\underset{\alpha}{\leftrightarrows}} (\mathbb{R}, \leq)$, with $\alpha(x) = \lfloor x \rfloor$ and $\gamma(y) = \lceil y \rceil$
3. $(\mathcal{P}(\mathbb{R}), \subseteq) \overset{\gamma}{\underset{\alpha}{\leftrightarrows}} (\mathcal{P}(\mathbb{R}), \subseteq)$ with $\alpha(X) = \{\lfloor x \rfloor \mid x \in X\}$ and $\gamma(Y) = \{\lceil y \rceil \mid y \in Y\}$
4. $(\mathcal{P}(\mathbb{R}), \subseteq) \overset{\gamma}{\underset{\alpha}{\leftrightarrows}} (\mathcal{P}(\mathbb{R}_0^+), \subseteq)$ with $\alpha(X) = \{|x| \mid x \in X\}$ and $\gamma(Y) = \{-y, y \mid y \in Y\}$, where $\mathbb{R}_0^+ = [0; \infty[$
5. $(\mathcal{P}(\mathbb{R}), \subseteq) \overset{\gamma}{\underset{\alpha}{\leftrightarrows}} (\mathcal{P}(\mathbb{R}), \subseteq)$ with $\alpha(X) = \{|x| \mid x \in X\}$ and $\gamma(Y) = \{-y, y \mid y \in Y\}$
6. $(\mathcal{P}(\mathbb{N}), \subseteq) \overset{\gamma}{\underset{\alpha}{\leftrightarrows}} (\mathcal{P}(\mathbb{N}), \subseteq)$ with $\alpha(X) = \emptyset$ and $\gamma(Y) = \mathbb{N}$
7. $(\mathcal{P}(\mathbb{N}), \subseteq) \overset{\gamma}{\underset{\alpha}{\leftrightarrows}} (\mathcal{P}(\mathbb{N}), \subseteq)$ with $\alpha(X) = \mathbb{N}$ and $\gamma(Y) = \emptyset$

## Problem 2: Monotone Functions (2 Points)

Assume that $f$ is a monotone function. Does $f(x_1) \leq f(x_2)$ imply $x_1 \leq x_2$? Give a proof or find a counterexample.

## Problem 3: Sign Analysis (3+3+4+4+7) Points)

In this exercise, we consider an analysis that can detect whether a variable is 0, $< 0$, $\leq 0$, $> 0$, or $\geq 0$.

1. Choose a suitable order $\sqsubseteq$, such that the set $S = \{0, < 0, \leq 0, > 0, \geq 0, \top, \bot\}$ is a complete lattice. Draw a Hasse diagram for the set.
2. Define an abstraction function $\alpha : \mathcal{P}(\mathbb{Z}) \to S$, and a concretization function $\gamma : S \to \mathcal{P}(\mathbb{Z})$, such that $(\mathcal{P}(\mathbb{Z}), \subseteq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} (S, \sqsubseteq)$ is a Galois connection.
3. Prove that $(\mathcal{P}(\mathbb{Z}), \subseteq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} (S, \sqsubseteq)$ is indeed a Galois connection.
4. Derive the abstract operators $+^{\#}$ and $-^{\#}$ for addition and subtraction:

| $+^{\#}$ | $\bot$ | $0$ | $< 0$ | $\leq 0$ | $> 0$ | $\geq 0$ | $\top$ |
|---|---|---|---|---|---|---|---|
| $\bot$ | | | | | | | |
| $0$ | | | | | | | |
| $< 0$ | | | | | | | |
| $\leq 0$ | | | | | | | |
| $> 0$ | | | | | | | |
| $\geq 0$ | | | | | | | |
| $\top$ | | | | | | | |

| $-^{\#}$ | $\bot$ | $0$ | $< 0$ | $\leq 0$ | $> 0$ | $\geq 0$ | $\top$ |
|---|---|---|---|---|---|---|---|
| $\bot$ | | | | | | | |
| $0$ | | | | | | | |
| $< 0$ | | | | | | | |
| $\leq 0$ | | | | | | | |
| $> 0$ | | | | | | | |
| $\geq 0$ | | | | | | | |
| $\top$ | | | | | | | |

5. Build the control-flow graph for the following program and perform a *Sign Analysis*, analogous to the *Abstract Reachability Analysis* shown in the last lecture.

```
x = 10
y = 2
z = 0
while (x > 0) {
    y = y+y
    if (x<y)
        y = 0
    else
        z = z-x
    x = x-1
}
```

## * Problem 4: Fixed points (4 Bonus Points)

A fixed point $x$ of a function $f$ is called *unique* if for every $y$ such that $f(y) = y$, $y = x$.

Prove that if there exists an $n \in \mathbb{N}^+$ such that $x$ is a unique fixed point of $f^n$, then $x$ is also a fixed point of $f$.