Assignment 3

Problem 1: Galois Connections (7 Points)

Which of the following examples are not Galois connections? Find counterexamples for those cases!

1. $\left(\mathbb{R}, \leq\right) \xrightarrow{\gamma} \left(\mathbb{R}, \leq\right)$, with $\alpha(x) = \lceil x \rceil$ and $\gamma(y) = \lfloor y \rfloor$
2. $\left(\mathbb{R}, \leq\right) \xrightarrow{\gamma} \left(\mathbb{R}, \leq\right)$, with $\alpha(x) = \lfloor x \rfloor$ and $\gamma(y) = \lceil y \rceil$
3. $\left(\mathcal{P}(\mathbb{R}), \subseteq\right) \xrightarrow{\gamma} \left(\mathcal{P}(\mathbb{R}), \subseteq\right)$ with $\alpha(X) = \{\lfloor x \rfloor \mid x \in X\}$ and $\gamma(Y) = \{\lceil y \rceil \mid y \in Y\}$
4. $\left(\mathcal{P}(\mathbb{R}), \subseteq\right) \xrightarrow{\gamma} \left(\mathcal{P}(\mathbb{R}_+), \subseteq\right)$ with $\alpha(X) = \{\lfloor x \rfloor \mid x \in X\}$ and $\gamma(Y) = \{-y, y \mid y \in Y\}$, where $\mathbb{R}_+ = [0; \infty[$
5. $\left(\mathcal{P}(\mathbb{R}), \subseteq\right) \xrightarrow{\gamma} \left(\mathcal{P}(\mathbb{R}), \subseteq\right)$ with $\alpha(X) = \{\lfloor x \rfloor \mid x \in X\}$ and $\gamma(Y) = \{-y, y \mid y \in Y\}$
6. $\left(\mathcal{P}(\mathbb{N}), \subseteq\right) \xrightarrow{\gamma} \left(\mathcal{P}(\mathbb{N}), \subseteq\right)$ with $\alpha(X) = \emptyset$ and $\gamma(Y) = \mathbb{N}$
7. $\left(\mathcal{P}(\mathbb{N}), \subseteq\right) \xrightarrow{\gamma} \left(\mathcal{P}(\mathbb{N}), \subseteq\right)$ with $\alpha(X) = \mathbb{N}$ and $\gamma(Y) = \emptyset$

Problem 2: Monotone Functions (2 Points)

Assume that $f$ is a monotone function. Does $f(x_1) \leq f(x_2)$ imply $x_1 \leq x_2$? Give a proof or find a counterexample.
Problem 3: Sign Analysis (3+3+4+4+7 Points)

In this exercise, we consider an analysis that can detect whether a variable is $0$, $<0$, $\leq 0$, $>0$, or $\geq 0$.

1. Choose a suitable order $\sqsubseteq$, such that the set $S = \{0, <0, \leq 0, >0, \geq 0, \top, \bot\}$ is a complete lattice. Draw a Hasse diagram for the set.

2. Define an abstraction function $\alpha : \mathcal{P}(\mathbb{Z}) \rightarrow S$, and a concretization function $\gamma : S \rightarrow \mathcal{P}(\mathbb{Z})$, such that $(\mathcal{P}(\mathbb{Z}), \subseteq) \xrightarrow{\gamma \circ \alpha} (S, \sqsubseteq)$ is a Galois connection.

3. Prove that $(\mathcal{P}(\mathbb{Z}), \subseteq) \xrightarrow{\gamma \circ \alpha} (S, \sqsubseteq)$ is indeed a Galois connection.

4. Derive the abstract operators $+\#$ and $-\#$ for addition and subtraction:

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<th>$\bot$</th>
<th>$0$</th>
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5. Build the control-flow graph for the following program and perform a Sign Analysis, analogous to the Abstract Reachability Analysis shown in the last lecture.

```
x = 10
y = 2
z = 0
while (x > 0) {
    y = y+y
    if (x<y)
        y = 0
    else
        z = z-x
    x = x-1
}
```

*Problem 4: Fixed points (4 Bonus Points)*

A fixed point $x$ of a function $f$ is called unique if for every $y$ such that $f(y) = y$, $y = x$. Prove that if there exists an $n \in \mathbb{N}^+$ such that $x$ is a unique fixed point of $f^n$, then $x$ is also a fixed point of $f$. 